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KINEMATICS OF GEOMAGNETIC SECULAR VARIATION IN A PERFECTLY CONDUCTING CORE

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CONTENTS

The motions in the Earth's electrically conducting fluid core which are the probable cause of the geomagnetic secular variations have time scales of the order of a few centuries or less. Seismic bounds on the kinematic molecular viscosity of the core and order-of-magnitude arguments about the eddy viscosity make plausible the hypothesis that at such short periods the core motion consists of a boundary layer of Ekman–Hartmann type close to the core mantle boundary, and an interior free-stream motion where the viscosity and resistivity can be set equal to zero. This boundary-layer approximation requires that the unknown vertical length scale of the poloidal geomagnetic field deep in the core be at least as long as the 600 km horizontal length scales inferred at the surface of the core from observations above the mantle. For periods shorter than a century the Ekman and magnetic boundary layers are probably thinner than 120 km.

If magnetic flux diffusion is neglected (i.e. if electrical conductivity is considered infinite) in the free stream in the core then the external geomagnetic field is completely determined by the fluid motion at the top of the free stream. Therefore the hypothesis of negligible flux diffusion in the free stream has implications for the geomagnetic secular variation, and these implications can be used as a test of whether there is *any* motion of a perfectly conducting core which will produce the observed secular variation. If the observed secular variation passes this test, we can write down explicitly all 'eligible' velocity fields, i.e. all velocity fields at the top of the free stream in the core which are capable of producing exactly the observed secular variation. The different eligible velocity fields are obtained by different choices of an arbitrary stream function on the surface of the core. We describe a method of selecting from among all eligible velocity fields those which are of particular geophysical interest, such as the one which is most nearly a rigid rotation (westward drift) or the one which is most nearly a latitude dependent westward drift with m degrees of freedom.

1. Introduction

This paper is an examination of what can be learned about the motions of the fluid near the surface of the Earth's core from the geomagnetic secular variations with time scales of a century or less. Secular variations with time scales shorter than three or four years are strongly attenuated by the conductivity of the mantle (McDonald 1957; Currie 1967), so the time scales of interest lie between 4 and 100 y. For longer time scales, flux diffusion is almost certainly important in the core, but Roberts & Scott (1965) argue that for the time scales considered here the core advects the geomagnetic field as a perfect conductor, except of course in thin boundary layers at the surface of the core. This picture attributes magnetic secular variations with time scales shorter than a century to magnetic flux advection by the motion of the core fluid in the free stream just below the boundary layers.

Allan & Bullard (1958, 1966) suggest that even on the decade time scale an important cause of secular variation is the upwelling of fluid near the surface of the core, with consequent ejection of magnetic flux by diffusion. The question at issue between Hide, Roberts and Scott and Allan & Bullard is whether in fact the toroidal field and the upwellings of fluid are sufficiently strong to produce the observed secular variation. In the absence of a good understanding of the dynamics of the core we might hope to examine this question empirically.

Kahle, Ball & Vestine (1967) have adopted the hypothesis of Hide, Roberts and Scott and have attempted to use the secular variation to determine the velocity of the fluid near the surface of the core. They point out that in principle the method cannot give the fine-scale structure of the velocity field, since the corresponding magnetic Reynolds numbers are too small to permit neglecting flux diffusion in the core. In any case the fine details of the magnetic field at the surface of the core cannot be obtained from information at the top of the mantle as long as experimental errors are not exactly zero. This latter problem is well understood. Consider a harmonic vector field without external sources, whose potential has the angular dependence of a spherical harmonic order l. If it has magnitude M at the surface of the earth, its magnitude at the core-mantle boundary is $M(a/b)^{l+2}$ where a is the outer and b the inner radius of the mantle. (Here we assume the mantle to be an insulator; mantle conductivity increases the amplification with depth.) Therefore, for any particular fractional error ϵ in the measurement of the geomagnetic field $\mathbf{B}$ outside the mantle, our direct observational information about $\mathbf{B}$ at the core-mantle boundary is confined to angular orders l which are less than the cutoff l_ϵ given roughly by

$$l_\epsilon + 2 \approx (\ln \epsilon^{-1})/\ln (a/b).$$

Within these limits Kahle *et al.* (1967) and Booker (1968) find that they can extrapolate $\mathbf{B}$ and $\partial_t \mathbf{B}$, the time derivative of $\mathbf{B}$, from the surface of the earth to the surface of the core. We shall assume that this extrapolation has been accomplished, recognizing that fine-scale detail is neither obtainable nor amenable to discussion in a perfectly conducting model of the core.

In the present paper, then, we assume that the geomagnetic field $\mathbf{B}$ and its secular variation, $\partial_t \mathbf{B}$, are observable just above the surface of the fluid core. We examine the viscous and magnetic boundary layers in the core at its boundary with the mantle, in order to learn how accurately our 'observed' values of **B** and $\partial_t \mathbf{B}$ just outside the core reflect the values at the top of the free stream, where we hope to get velocity information. Assuming that the fluid in the free stream is a perfect conductor, we deduce relations which must hold between **B** and $\partial_t \mathbf{B}$ if the velocity of the free stream is continuous. Failure of these relations means that $\partial_t \mathbf{B}$ cannot be obtained from **B** by any continuous motion of a perfectly conducting fluid; flux diffusion must be important. If **B** and $\partial_t \mathbf{B}$ pass these tests then there is a very large class of 'eligible' velocity fields **v** at the top of the free stream in the core which are continuous and can exactly reproduce $\partial_t \mathbf{B}$ from **B**.

Thus the goal which Kahle *et al.* have set themselves is in principle unattainable. We show that it is, nonetheless, possible to use the observations of secular variation to answer questions about the fluid velocity at the top of the free stream in the core, if we accept the hypothesis of Hide, Roberts and Scott and frame the questions in a manner which takes explicit account of the multiplicity of eligible velocity fields.

2. Length scales too short for perfect conductivity

It is well known (see, for example, Backus 1958) that the magnetic field produced by electric currents in a sphere of radius R is the sum of a toroidal field which vanishes outside the sphere and a poloidal field which does not. The poloidal field is a superposition of multipole fields $\mathbf{P}_{lmn}$ which have the form

$$\mathbf{P}_{lmn} = \nabla \times (\mathbf{r} \times \nabla p_{lmn}).$$

When $r < R$ the scalar is
$$p_{lmn} = j_l(\alpha_{l-1,n} r/R)\; Y_l^m(\theta, \lambda), \tag{1}$$

Here $\mathbf{r}$ is the radius vector from the centre of the sphere, and r, θ, λ are radius, colatitude, and longitude; Y_l^m is the surface spherical harmonic of total angular order l with longitude dependence $e^{i\lambda m}$, j_l is the spherical Bessel function of order l, and $\alpha_{l,n}$ is its nth positive zero.

Mie (1908) has shown that in a rigid sphere with constant electrical conductivity σ and radius R the poloidal and toroidal multipole modes decay independently and exponentially with mean lives independent of m. The mean life of mode (1) is

$$T_{ln} = \mu_0 \sigma R^2/(\alpha_{l-1,n})^2$$

where μ_0 is the magnetic permeability of free space.

If we expand the geomagnetic poloidal field in the form

$$\mathbf{P} = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} A_{lmn}(t)\, \mathbf{P}_{lmn}(r, \theta, \lambda)$$

then
$$\frac{\mathrm{d}A_{lmn}}{\mathrm{d}t} = -\frac{A_{lmn}}{T_{ln}} + f_{lmn},$$

where the free-decay term, A_{lmn}/T_{ln}, represents flux diffusion out of the mode $\mathbf{P}_{lmn}$, while the term f_{lmn}, which vanishes for a rigid core, represents transfer of flux into the mode $\mathbf{P}_{lmn}$ by the velocity field (Elsasser 1946). In discussing the behaviour of $A_{lmn}(t)$ over time intervals of length T we certainly cannot neglect flux diffusion if $T_{ln} \leqslant T$. If we are to treat the core as a perfect conductor for a period as long as 100 y we must restrict attention to those $\mathbf{P}_{lmn}$ which have $T_{ln} \geqslant 300$ y, or perhaps $T_{ln} \geqslant 1000$ y.

After a review of the literature, Tozer (1958) reports that $\sigma = 3\times10^5$ mho/m within a factor of 3. We adopt Tozer's estimate throughout this paper. Then $T_{ln} \geqslant 1000$ years requires $\alpha_{l-1,n} \leqslant 11{\cdot}8$, while $T_{ln} \geqslant 300$ y requires $\alpha_{l-1,n} \leqslant 21{\cdot}5$. Figure 1 shows these two admissible sets of modes.

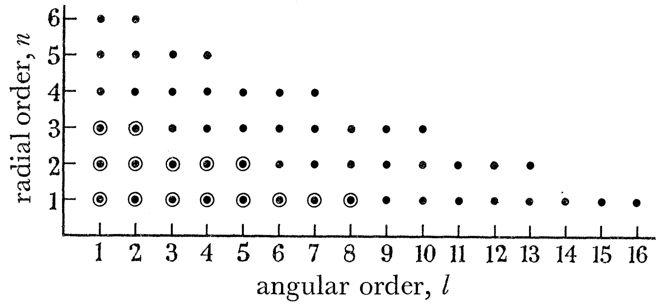

FIGURE 1. Values of total angular order l and radial order n such that the free decay time of the geomagnetic poloidal mode $\mathbf{P}_{lmn}$ is at least 300 years. Circled dots have a decay time of at least 1000 years. The electrical conductivity of the core is taken as 3×10^5 mho/m.

How serious are the restrictions implied by figure 1? Of course we have no direct information about the variation of $\mathbf{B}$ with r in the core, so we cannot assess the importance of the limitation on n. The limitation on l can be compared with the amplitudes of the spherical harmonics in the external field. Cain, Hendricks, Langell & Hudson (1967) have obtained those amplitudes up to $l = 10$ from surface and satellite observations. Extrapolating these harmonics to the core-mantle boundary on the assumption that the mantle is an insulator, Booker (1968) finds that most of the horizontal variation in $\mathbf{B}$ for $l \leqslant 10$ on the surface of the core is accounted for by angular orders with $l \leqslant 6$. Therefore figure 1 does not rule out the possibility of treating the core as a perfect conductor in a discussion of the Fourier amplitudes of the secular variation with periods shorter than 100 years.

On the surface of the core the angular order $l = 6$ corresponds to a horizontal length scale L_H of about one sixth of the core radius. i.e. about 600 km.

The necessary conditions for neglecting flux diffusion, summarized in figure 1, are by no means sufficient. Allan & Bullard (1958, 1966) have shown that sufficiently rapid upwelling of fluid in a sufficiently strong toroidal field can eject flux by diffusion rapidly enough to explain the whole secular variation. Cox & Dalrymple (1967) find that the main field probably reverses in less than 5000 y, passing close to zero in the process; this reversal can be accomplished only by flux diffusion and requires that in the core the main dipole field have a higher radial mode than $n = 1$. Dagley *et al.* (1967) find evidence that the reversal time may be 15000 years. The present paper is an examination of what can be said about the secular variation between reversals if in fact flux diffusion is then negligible in the free stream of the core for periods shorter than a century.

3. THE BOUNDARY LAYERS

The core model we consider consists of a free-stream motion with a length scale $L = 600$ km or greater, in which the fluid behaves as if perfectly conducting and non-viscous; and in addition thin layers at the core-mantle boundary where the motion of the free stream is modified by resistivity and viscosity to fit the boundary conditions which the free stream cannot meet. In subsequent sections we examine the free stream. In the present section we intend to study the boundary layers and in particular to verify *a posteriori* that they are considerably thinner than L. We restrict attention to those Fourier components of the secular variation which have periods between 4 and 100 y. They are slow enough to penetrate the mantle and fast enough that figure 1 does not refute our model at the outset.

Greenspan & Howard (1963) have shown that in addition to boundary layers, viscosity produces a small 'spin-up' throughout the core (a transfer of angular momentum by pressure-driven advection). Since we do not examine the dynamics of the free stream we can regard this spin-up as part of the free-stream motion.

Proceeding as usual in a boundary layer theory, we choose a quasi-cubical region Q one of whose faces is part of the core mantle boundary and whose diameter is much larger than the boundary layer thicknesses and much less than L. Then in Q the core/mantle boundary is nearly plane and the free-stream velocity and magnetic field are nearly constant, while the velocity and magnetic field in the boundary layers are functions mainly of time and the normal distance from the boundary. To study these boundary layers we assume we know the free-stream motion, and we introduce an accelerated reference frame which moves with the free stream in Q. In this frame we have a hydromagnetic Rayleigh problem: what is the motion of a fluid of density ρ, kinematic viscosity ν, and magnetic diffusivity $\kappa = (\mu_0 \sigma)^{-1}$ which fills the three-dimensional half-space $z \geqslant 0$, if the fluid is at rest at $z = +\infty$ and the x-y plane is a rigid boundary moving tangentially in a known fashion. A magnetic field is present which is constant at $z = +\infty$ and interacts with the fluid motion. We neglect compressibility in the core.

To solve Rayleigh's problem we write the total magnetic field as

$$\mathbf{B} = \mathbf{B}_0 + (\hat{\mathbf{z}}\,.\,\mathbf{B}_0)\,\mathbf{b}(z, t)$$

where $\mathbf{B}_0$ is constant, $\hat{\mathbf{z}}$ is the unit vector in the direction of increasing z, and $\mathbf{b}$ is a dimensionless form of the perturbation in the magnetic field produced by the motion of the boundary. We define

$$A = (\hat{\mathbf{z}}\,.\,\mathbf{B}_0)\,(\mu_0 \rho)^{-\frac{1}{2}},$$

the Alfvén speed normal to the boundary, and we write the fluid velocity as $A\mathbf{v}(z, t)$. We assume that $\mathbf{b}$ and $\mathbf{v}$ are independent of x and y. Then from $\nabla\,.\,\mathbf{v} = 0$ and $\nabla\,.\,\mathbf{b} = 0$ we deduce that $\hat{\mathbf{z}}\,.\,\mathbf{v}$ and $\hat{\mathbf{z}}\,.\,\mathbf{b}$ are independent of z. The boundary condition $\hat{\mathbf{z}}\,.\,\mathbf{v} = 0$ at $z = 0$ and, if necessary, and appropriate redefinition of $\mathbf{B}_0$, permit us to infer that $\mathbf{b}\,.\,\hat{\mathbf{z}}$ and $\mathbf{v}\,.\,\hat{\mathbf{z}}$ vanish for all positive z at all times.

We denote by f the local Coriolis parameter, $2\hat{\mathbf{z}}\,.\,\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the angular velocity of rotation of the Earth. Then $f = 2\Omega\cos\theta$ where θ is geographic colatitude.

In the simple Rayleigh geometry the exact equations of motion and magnetic variation are

$$\left.\begin{aligned} \partial_t \mathbf{v} + f\hat{\mathbf{z}} \times \mathbf{v} &= A\partial_z \mathbf{b} + \nu\,\partial_z^2 \mathbf{v}, \\ \partial_t \mathbf{b} &= A\,\partial_z \mathbf{v} + \kappa\,\partial_z^2 \mathbf{b}. \end{aligned}\right\} \tag{2}$$

Following the usual procedure in boundary layer theory, we want to solve (2) for $\mathbf{v}$ and $\mathbf{b}$ using boundary conditions obtained from the free stream. We consider only the steady problem, where the time variations are of the form $e^{-i\omega t}$, with ω real and constant.

We need four vector boundary conditions to specify a solution of (2). Two of these we already have: $\mathbf{b}$ and $\mathbf{v}$ must vanish at $z = +\infty$. A third condition is that at $z = 0$ the fluid velocity $A\mathbf{v}$ must be the velocity of the rigid boundary relative to the free stream, a velocity which, in this boundary layer calculation, we assume is known. What is the fourth vector boundary condition?

To find it we must re-examine the whole free-stream problem in the spherical configuration. We assume that the free-stream angular frequencies ω under discussion are small enough to permit treating the mantle as an insulator. Let V be the spherical (or topologically spherical) region occupied by the perfectly conducting fluid. Let ∂V be its boundary and let $\hat{\mathbf{n}}$ be the outward unit normal to ∂V. (Thus $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ in the boundary layer problem.) Inside V the magnetic field at any fluid point is completely determined by its initial value there and by the spatial gradient of the fluid displacement at that point (Cowling 1957). From Maxwell's equations, $\hat{\mathbf{n}}.\mathbf{B}$ must be continuous across ∂V. Outside V, $\mathbf{B}$ is the gradient of a potential which vanishes at infinity. To determine this potential from $\hat{\mathbf{n}}.\mathbf{B}$ on ∂V is an exterior Neumann problem, which has a unique solution. In fine, the fluid displacement determines $\hat{\mathbf{n}}\times\mathbf{B}$ just outside and just inside ∂V in quite different ways, and a general displacement of the fluid will usually produce a discontinuity in $\hat{\mathbf{n}}\times\mathbf{B}$ at ∂V. This jump (outer value minus inner value) in $\hat{\mathbf{n}}\times\mathbf{B}$ is the value which must be assigned to $(\mathbf{B}_0.\hat{\mathbf{z}})\,\mathbf{b}$ at $z = 0$ in solving (2). It is our fourth vector boundary condition and, unlike the other three, is obtained from the global rather than the local properties of the free stream.

We will need a rough estimate of the relative sizes of $\mathbf{b}(0)$ and $\mathbf{v}(0)$. The foregoing paragraphs shows that $(\mathbf{B}_0.\hat{\mathbf{z}})\,\mathbf{b}(0)$ is of the same order of magnitude as the change produced in the free-stream value of $\hat{\mathbf{n}}.\mathbf{B}$ on ∂V by the free-stream fluid displacement there. For a free-stream Fourier component with period $2\pi/|\omega|$ seconds, this displacement is $A\mathbf{v}(0)/|\omega|$, and it produces in $\hat{\mathbf{n}}.\mathbf{B}$ a change $C'|\omega|^{-1}A\mathbf{v}(0).\nabla_S(\hat{\mathbf{n}}.\mathbf{B})$ where C' is a dimensionless number of the order of or less than unity. Here ∇_S is the tangential gradient operator on ∂V: $\nabla_S = \nabla-\hat{\mathbf{n}}(\hat{\mathbf{n}}.\nabla)$. Thus

$$(\mathbf{B}_0.\hat{\mathbf{z}})\,\mathbf{b}(0) \approx C''|\omega|^{-1}A\mathbf{v}(0).\nabla_S(\hat{\mathbf{n}}.\mathbf{B}),$$

where C'' is another dimensionless constant less than or of the order of 1. But

$$\mathbf{B}_0.\hat{\mathbf{z}} = -\hat{\mathbf{n}}.\mathbf{B} \quad \text{and} \quad |\nabla_S(\hat{\mathbf{n}}.\mathbf{B})/(\hat{\mathbf{n}}.\mathbf{B})| \approx L_H^{-1},$$

where L_H is the horizontal length scale of the free stream. Thus,

$$|\mathbf{b}(0)| = C\left|\frac{A}{\omega L_H}\right||\mathbf{v}(0)|, \tag{3}$$

where $A\mathbf{v}(0)$ and $(\mathbf{B}_0.\hat{\mathbf{z}})\,\mathbf{b}(0)$ are the boundary values in the Rayleigh problem (2), and therefore the velocity and the discontinuity in $\hat{\mathbf{n}}\times\mathbf{B}$ produced on ∂V by the free-stream motion. The dimensionless constant C is of the order of or less than unity, and is a sort of shape factor for the displacement. If the displacement is a pure rigid rotation of the whole fluid, C vanishes (Backus 1958).

Roberts & Scott (1965) argue that the Lorentz force on the surface current entailed by a discontinuity in $\hat{\mathbf{n}}\times\mathbf{B}$ at ∂V will alter the free-stream motion so as to eliminate the discontinuity, so that we must take $\mathbf{b}(0) = \mathbf{0}$ in solving (2). I believe that their argument requires values of $\hat{\mathbf{n}}.\mathbf{B}$ considerably larger than those present in the Earth's core, but a discussion is postponed to Appendix II because in either case we now have enough boundary conditions to solve (2).

Hide & Roberts (1960) solve (2) for all possible values of the parameters. For the earth's core we can specialize and simplify the discussion slightly. We consider a Fourier component of the free-stream motion whose time variation is $e^{-i\omega t}$ where ω is a real positive or negative constant such that $2\pi/|\omega|$ lies between 4 and 100 y. Equations (2) are four equations for the four unknown complex components v_x, v_y, b_x, b_y. We define four new variables, v_+, v_-, b_+, b_- as follows:

$$v_\tau = v_x + i\tau v_y,$$
$$b_\tau = v_x + i\tau b_y,$$

with $\tau = +1$ or -1. Then the four equations (2) decouple into two pairs, one pair for $\tau = +1$ and one pair for $\tau = -1$:

$$[\nu\,\partial_z^2 + i(\omega - \tau f)]\,v_\tau + A\,\partial_z b_\tau = 0,$$
$$A\,\partial_z v_\tau + [\kappa\,\partial_z^2 + i\omega]\,b_\tau = 0.$$

For each fixed τ, any solution b_τ, v_τ is a linear combination of four particular solutions with z-dependence $\exp(-n_\tau z)$. The four complex decay constants n_τ are the four roots of the secular equation

$$(n_\tau)^4 - (n_\tau)^2\left[\frac{A^2}{\kappa\nu} + i\left(\frac{\tau f - \omega}{\nu}\right) - i\left(\frac{\omega}{\kappa}\right)\right] + \frac{\omega(\tau f - \omega)}{\kappa\nu} = 0.$$

We are interested only in the two roots with positive real part, because our solutions b_τ, v_τ must vanish at $z = +\infty$. We label these two roots as $n_{\nu\tau}$ and $n_{\kappa\tau}$; the rule for assigning the labels ν and κ will be specified later.

To summarize, for each fixed angular frequency ω there are associated with (2) four complex constants, $n_{\nu+}$, $n_{\nu-}$, $n_{\kappa+}$, $n_{\kappa-}$, and to each of these constants corresponds a boundary layer. The thicknesses of the four boundary layers are $(\mathscr{R}n_{\nu+})^{-1}$, $(\mathscr{R}n_{\nu-})^{-1}$, $(\mathscr{R}n_{\kappa+})^{-1}$ and $(\mathscr{R}n_{\kappa-})^{-1}$ where $\mathscr{R}$ denotes real part. The two boundary layers corresponding to $n_{\nu+}$ and $n_{\nu-}$ will be called Ekman layers, while those corresponding to $n_{\kappa+}$ and $n_{\kappa-}$ will be called magnetic boundary layers. The question at issue is whether all four of these boundary layer thicknesses are considerably less than L, the length scale of the free stream.

To examine this question we introduce dimensionless variables:

$$\alpha_\tau = \frac{A^2}{\kappa|\tau f - \omega|}, \quad \beta_\tau = \frac{\kappa(\tau f - \omega)}{\nu\omega}, \quad \gamma = \frac{A^2}{\nu|\omega|} = \alpha_\tau|\beta_\tau|.$$

We define

$$m_\tau = \left|\frac{\kappa}{\omega}\right|^{\frac{1}{2}} n_\tau$$

if $\omega > 0$, and

$$m_\tau = \left|\frac{\kappa}{\omega}\right|^{\frac{1}{2}} n_\tau^*$$

if $\omega < 0$; here n^* means the complex conjugate of n. Then the secular equation becomes

$$(m_\tau)^4 - (m_\tau)^2\,[\gamma + i(\beta_\tau - 1)] + \beta_\tau = 0. \tag{4}$$

If boundary layer theory is to be applicable, all four of the boundary layer thicknesses should be less than some small fraction ζ of the free-stream length scale L. We will suppose that $\zeta \leqslant 0{\cdot}2$ suffices, although without a theory of the free stream it is impossible to be sure of this. The demand that all four boundary layers be thinner than ζL can be restated as follows: for either $\tau = +1$ or $\tau = -1$, all four of the roots m_τ of (4) must satisfy

$$|\mathscr{R} m_\tau| > u,$$

where
$$u = \frac{1}{\zeta L}\left|\frac{\kappa}{\omega}\right|^{\frac{1}{2}}. \tag{5}$$

With $\zeta = 0{\cdot}2$, $L = 600$ km, $\kappa = 2{\cdot}5 \times 10^4$ cm²/s, and $T = 2\pi/|\omega|$ we have $u = 0{\cdot}3\,T^{\frac{1}{2}}$ when T is measured in centuries. Routine but laborious algebra, given in appendix I, shows that if $u \leqslant 0{\cdot}3$ then all roots of (4) have $|\mathscr{R} m_\tau| > u$ whenever both of the following conditions are satisfied:

$$\alpha_\tau \leqslant 0{\cdot}45 u^{-\frac{2}{3}}, \tag{6a}$$

$$|\beta_\tau| \geqslant 4u^2. \tag{6b}$$

In our problem, $100|\omega| \leqslant |f|$ except within 2° of the geographical equator. Outside this narrow equatorial strip we commit an error smaller than 1 % by writing

$$\alpha_\tau = \frac{A^2}{\kappa |f|}, \quad \beta_\tau = \frac{\tau\kappa f}{\omega\nu}.$$

With this approximation conditions (6) become

$$|A| \leqslant 0{\cdot}67 |f|^{\frac{1}{2}} |\omega|^{\frac{1}{6}} |\kappa \zeta L|^{\frac{1}{3}} \text{ cm/s},$$

$$\nu \leqslant \tfrac{1}{4} L^2 \zeta^2 |f| \text{ cm}^2/\text{s}.$$

Substituting $L = 600$ km, $\zeta = 0{\cdot}2$, $\kappa = 2{\cdot}5 \times 10^4$ cm²/s and $2\pi/|\omega| = T$ centuries, we have

$$|A| \leqslant 1{\cdot}9\,T^{-\frac{1}{6}}\,|\cos\theta|^{\frac{1}{2}} \text{ cm/s}, \tag{7a}$$

$$\nu \leqslant 5{\cdot}2 \times 10^9 |\cos\theta| \text{ cm}^2/\text{s}. \tag{7b}$$

Because $u \leqslant 0{\cdot}3$, conditions (7) suffice to insure that all four boundary layers will be thinner than ζL, i.e. 120 km. There remains the task of estimating ν and A at the surface of the core.

From Cain's (1967) spherical harmonic amplitudes of the geomagnetic field up to angular order $l = 10$, Booker (1968) finds that on the surface of the core $\hat{\mathbf{n}}.\mathbf{B}$ nowhere exceeds 5 G, so $|A| \leqslant 0{\cdot}45$ cm/s. Thus if $T \leqslant 1$ century, (7 *a*) is satisfied except within 3° of the geographic equator. The failure of (7 *a*) at the geographic equator lets one of the magnetic boundary layers there become infinitely thick. This may invalidate the whole boundary layer approximation, but we think that unlikely. In the mathematically analogous problem of the steady flow of a viscous, conducting fluid through a cylindrical insulating pipe in the presence of a constant transverse magnetic field, Roberts (1967) has shown that the boundary layer is the usual Hartmann layer except at the singularities where the Hartmann layer thickness becomes infinite. Near these singularities a boundary layer of parabolic type is required, but it is still a boundary layer, and does not invalidate the Hartmann-layer solution where the latter exists.

The kinematic viscosity of the core has been estimated by Jeffreys (1959) from the passage of seismic waves through the core without detectable damping. Gutenberg's (Hide 1956) correction makes Jeffreys's bound $\nu \leqslant 10^9$ cm²/s . Even with this very weak bound on ν, condition (7 *b*) is satisfied except within 11° of the equator.

It seems possible that ν in the core is very much less than Jeffreys's bound. Nachtrieb & Petit (1956) fit their data for liquid mercury below 10 kb and 400° Kelvin with an equation of the form

$$\ln\left(\frac{\nu}{\nu_0}\right) = -(3g+2)\left[1-\left(\frac{\rho_0}{\rho}\right)^{\frac{1}{3}}\right]+\left(\frac{E^*+pV^*}{RT}\right)-\left(\frac{E^*+p_0V^*}{RT_0}\right), \tag{8}$$

where R is the gas constant, g is Grüneisen's constant, E^* is an activation energy of $4{\cdot}21\times10^{10}$ ergs mole^{-1} °K, V^* is an activation volume of 0·885 cm^3/mole (Nachtrieb & Petit give $V^* = 0{\cdot}587$ cm^3/mole, but they seem to have made an arithmetic error in reducing their published data), and subscript zero denotes a reference state.

It appears that no measurements of the viscosity of iron even at kilobar pressures have been made, so E^* and V^* for iron are unknown. Suppose we assume, without experimental or theoretical justification, that (8) holds for iron at 2 Mb and several thousand degrees Kelvin, with roughly the E^* and V^* measured by Nachtrieb & Petit below 10 kb. For iron, $g = 1{\cdot}6$, while at the top of the core $\rho = 9{\cdot}7$ g/cm^3 and $p = 1{\cdot}8$ Mb (Bullen 1963). If we take for the reference state $p_0 = 1$ bar, $T_0 = 1700$ °K, then ν_0 for iron is about 10^{-2} cm^2/s and ρ_0 is 6·9 g/cm^3. Equation (8) becomes

$$\log_{10}\nu \approx \frac{8{\cdot}3\times10^3}{T}-2{\cdot}5.$$

If $T = 2000$ °K, this naive estimate predicts $\nu \approx 50$ cm^2/s at the top of the core. The estimate of ν is considerably more sensitive to V^* than to the other parameters, and the difference between iron and mercury here could be very serious indeed. In addition, Cohen & Turnbull (1959) give a qualitative argument that in (8) V^* should increase at high pressure. The evidence that $\nu \ll 10^9$ cm^2/s in the core is at best suggestive, but what little evidence there is points in this direction. If it is true, then (7 *b*) indicates that $|\beta| \gg 1$ is a very good approximation except in a thin strip at the geographic equator. In fact if $\nu = 10^4$ cm^2/s, for example, and if $|\alpha| \leqslant 1$, then the Ekman layer thicknesses are of the order of $|2\nu/f|^{\frac{1}{2}}$ while the magnetic boundary layer thicknesses are of the order of $|2\kappa/\omega|^{\frac{1}{2}}$. The former is 120 $|\cos\theta|^{-\frac{1}{2}}$ metres, while the latter is 70 km.

The referee correctly points out that quite possibly the relevant viscosity in (7 *b*) is an eddy viscosity. Attempts to estimate an eddy viscosity would constitute a separate and, in my opinion, non-trivial investigation. The purpose of the discussion of molecular viscosity given here is to show that the bounds on ν are poorer than commonly believed, but still good enough to justify boundary layer theory in the absence of an eddy viscosity so large as to violate (7 *b*). An eddy viscosity of 10^8 cm^2/s with velocities less than 0·5 cm/s requires eddies 4000 km across. This seems unlikely in a boundary layer, but the problem of turbulence in the core is a very difficult one.

In the remainder of this paper we will treat the core, V, as if it were a perfect conductor. The 'surface' of the core, ∂V, will refer to the top of the free stream, just below the Ekman and magnetic boundary layers.

4. Formulation of the problem for a perfectly conducting core

We assume that we have a simply connected volume V of perfectly conducting fluid in vacuum. Frozen into the fluid is an internal magnetic field, and magnetic flux penetrates the surface to produce an external magnetic field. The fluid moves with a velocity **v** so slow that the displacement current outside V is negligible. We assume that we can measure $\mathbf{B}$ just outside the boundary ∂V of V, but that we have no other information about $\mathbf{v}$. We denote by $\hat{\mathbf{n}}$ the unit outward normal on ∂V. What can we learn about $\mathbf{v}$?

At the outset it is clear that the external $\mathbf{B}$ is completely determined by the motion of the surface fluid ∂V, and does not directly reflect the motion inside V (Bondi & Gold 1950). Different internal motions which produce the same surface motion produce the same external $\mathbf{B}$. Therefore at best we can hope to determine $\mathbf{v}$ on ∂V. Because we are interested primarily in the earth's core, we assume that $\hat{\mathbf{n}}.\mathbf{v} = 0$ on ∂V.

The equation governing the magnetic field in a perfect conductor moving with velocity $\mathbf{v}$ is

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{9}$$

We observe $\mathbf{B}$ just outside ∂V. Since $\hat{\mathbf{n}}.\mathbf{B}$ must be continuous across ∂V, we know $\hat{\mathbf{n}}.\mathbf{B}$ in the fluid just inside ∂V, where (9) applies. If $\hat{\mathbf{n}} \times \mathbf{B}$ is also continuous across ∂V then we know $\hat{\mathbf{n}} \times \mathbf{B}$ just inside the fluid. We examine both cases, where we know only $\hat{\mathbf{n}}.\mathbf{B}$ and where we also know $\hat{\mathbf{n}} \times \mathbf{B}$ in the fluid just inside ∂V. What can we learn from equation (9) about $\mathbf{v}$ on ∂V?

5. Calculations in curvilinear coordinates

Even if ∂V is a sphere, it will be necessary to discuss (9) in curvilinear coordinate systems more general than ordinary spherical polar coordinates. The coordinate systems we are about to discuss have been treated at some length by Backus (1967) when ∂V is a sphere.

Let x^1 and x^2 be curvilinear coordinates on a patch (a relatively open subset) of ∂V. Let $\mathbf{r}(x^1, x^2)$ be the position vector of the point on ∂V whose coordinates are x^1, x^2. Let ∂_i denote the partial derivative with respect to x^i. We assume $\partial_1 \mathbf{r} \times \partial_2 \mathbf{r} \neq 0$. Let $\hat{\mathbf{n}}(x^1, x^2)$ be the unit outward normal to ∂V at $\mathbf{r}(x^1, x^2)$. Let ∇_S denote the surface gradient operator on ∂V

$$\nabla_S = \nabla - \hat{\mathbf{n}}(\hat{\mathbf{n}}.\nabla). \tag{10}$$

Let g_{ij} and g^{ij} be the covariant and contravariant components of the metric tensor on ∂V:

$$g_{ij} = \partial_i \mathbf{r} . \partial_j \mathbf{r},$$
$$g^{ij} = \nabla_S x^i . \nabla_S x^j.$$

Let f_{ij} be the covariant components of the second fundamental form on ∂V

$$f_{ij} = \hat{\mathbf{n}} . \partial_i \partial_j \mathbf{r}.$$

Let $f_i{}^j = f_{ik} g^{kj}$. Let D_i denote covariant differentiation with respect to x^i on ∂V.

We incorporate x^1 and x^2 as two of the coordinates in a three-dimensional system of curvilinear coordinates x^1, x^2, x^3 defined near ∂V. The coordinates x^1, x^2, x^3 are assigned to the point whose position vector is

$$\mathbf{R}(x^1, x^2, x^3) = \mathbf{r}(x^1, x^2) + x^3 \hat{\mathbf{n}}(x^1, x^2). \tag{11}$$

The equation of ∂V is $x^3 = 0$. We use the Einstein index conventions with greek indices taking the values, 1, 2, 3 while italic indices are 1, or 2. The covariant and contravariant components of the metric tensor in the coordinate system x^1, x^2, x^3 are

$$G_{\alpha\beta} = \partial_\alpha \mathbf{R} . \partial_\beta \mathbf{R},$$
$$G^{\alpha\beta} = \nabla x^\alpha . \nabla x^\beta,$$

so $G_{ij} = g_{ij}$, $G_{3j} = 0$, $G_{33} = 1$, $G^{ij} = g^{ij}$, $G^{3j} = 0$ and $G^{33} = 1$.

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Any vector field $\mathbf{Q}$ defined in 3-space can be written in terms of its covariant components Q_α as

$$\mathbf{Q} = Q_\alpha \nabla x^\alpha$$

and in terms of its contravariant components Q^α as

$$\mathbf{Q} = Q^\alpha \partial_\alpha \mathbf{R}.$$

On ∂V, Q^i and Q_i are the contravariant and covariant components of the tangent-vector field $\mathbf{Q}-\hat{\mathbf{n}}(\hat{\mathbf{n}}.\mathbf{Q})$, while $Q^3 = Q_3 = \hat{\mathbf{n}}.\mathbf{Q}$. Under a change of the coordinates x^1 and x^2 on ∂V, the component Q^3 does not change; its scalar behaviour prompts the definition $D_i Q^3 = \partial_i Q^3$.

Using the relations between $G_{\alpha\beta}$ and $g_{\alpha\beta}$ we can deduce the relations between the three-dimensional covariant derivatives $\mathscr{D}_\alpha$ and the surface covariant derivatives D_i. They are

$$\left.\begin{aligned} \mathscr{D}_i Q^j &= D_i Q^j - f_i^{\ j} Q^3, \\ \mathscr{D}_i Q^3 &= D_i Q^3 + f_{ij} Q^j, \\ \mathscr{D}_3 Q^i &= \partial_3 Q^i - f_k^{\ i} Q^k, \\ \mathscr{D}_3 Q^3 &= \partial_3 Q^3. \end{aligned}\right\} \tag{12}$$

Since $\nabla.\mathbf{B} = 0$, the contravariant components of equation (9) are

$$\partial_t B^\alpha = B^\beta \mathscr{D}_\beta v^\alpha - v^\beta \mathscr{D}_\beta B^\alpha - (\mathscr{D}_\beta v^\beta)\, B^\alpha. \tag{13}$$

If we assume $v^3 = 0$ and appeal to (12), equations (13) become

$$\partial_t B^3 + D_i(B^3 v^i) = 0, \tag{14}$$

$$\partial_t B^i = B^j D_j v^i - v^j D_j B^i + B^3 \partial_3 v^i - (\nabla.\mathbf{v})\, B^i. \tag{15}$$

Alternatively, these equations can be obtained simply by observing that from the symmetry of the Christoffel symbols

$$B^\beta \mathscr{D}_\beta v^\alpha - v^\beta \mathscr{D}_\beta B^\alpha = B^\beta \partial_\beta v^\alpha - v^\beta \partial_\beta B^\alpha,$$

$$B^j D_j v^i - v^j D_j B^i = B^j \partial_j v^i - v^j \partial_j B^i.$$

Roberts & Scott (1965) have already obtained what amounts to (14) when ∂V is a sphere and x^1 and x^2 are colatitude and longitude. Equation (14) is exactly the equation for the conservation of flux per unit area (B^3) in a two-dimensional fluid moving with tangential velocity $\mathbf{v}$ on ∂V.

We note in passing that if ∂V is a sphere the generalization of (14) to the case of $\kappa \neq 0$ can be obtained immediately from equation (62*a*) of Backus (1958) by observing that on ∂V we have $v_r = 0$ and $\partial_r v_r + D_i v^i = 0$. That generalization is

$$\partial_t B^3 + D_i(B^3 v^i) = \frac{\kappa}{r} \nabla^2 (r B^3)$$

where r is radial distance from the centre of the sphere.

6. Velocity information from $\hat{\mathbf{n}}.\mathbf{B}$

In this section we learn what we can about $\mathbf{v}$ from (14) when B^3 and $\partial_t B^3$ are known on ∂V but B^1 and B^2 just inside the fluid are not known, so that (15) is of no use.

In Gibbs's dyadic notation (14) can be given the suggestive form

$$\partial_t B^3 + \nabla_S.(B^3\mathbf{v}) = 0 \tag{16}$$

where the surface divergence $\nabla_S.\mathbf{q}$ is defined as $D_i q^i$. Kahle *et al.* (1967) try to find $\mathbf{v}$ uniquely by truncating its expansion in surface vector spherical harmonics and choosing the expansion coefficients to minimize the surface integral of the square of the left-hand side of (16) when the observations for B^3 and $\partial_t B^3$ are inserted there. We will see that if there is even one $\mathbf{v}$ which satisfies (16) there are infinitely many, and we will explicitly list them all.

By hypothesis ∂V is simply connected. Then if $\mathbf{q}$ is any vector field defined on and tangent to ∂V there are two uniquely determined scalars ϕ and ψ on ∂V such that

$$\int_{\partial V}\phi\,\mathrm{d}A = \int_{\partial V}\psi\,\mathrm{d}A = 0 \tag{17}$$

and

$$\mathbf{q} = \nabla_S\phi - \hat{\mathbf{n}}\times\nabla_S\psi. \tag{18}$$

For references and a discussion of this theorem see, for example, Backus (1966). It is easy to calculate that for any ϕ and ψ

$$\nabla_S.(\hat{\mathbf{n}}\times\nabla_S\psi) = 0 \tag{19}$$

and

$$\hat{\mathbf{n}}.[\nabla_S\times(\nabla_S\phi)] = 0;$$

therefore $\nabla_S\phi$ in (18) is called the irrotational part of $\mathbf{q}$ while $-\hat{\mathbf{n}}\times\nabla_S\psi$ is called the solenoidal part. The theorem just quoted can be rephrased thus: on any smooth surface topologically a sphere, any tangent vector field has a unique resolution into an irrotational and a solenoidal part.

We apply this theorem to $B^3\mathbf{v}$. We let ϕ and ψ be the scalars satisfying (17) and

$$B^3\mathbf{v} = \nabla_S\phi - \hat{\mathbf{n}}\times\nabla_S\psi. \tag{20}$$

Then, from (19), we see that (16) is simply

$$\partial_t B^3 + \nabla_S^2\phi = 0, \tag{21}$$

where ∇_S^2 means $\nabla_S.\nabla_S$ or $g^{ij}D_iD_j$. If ∂V is a sphere of radius R, ∇_S^2 is the angular part of the Laplacian:

$$R^2\nabla_S^2 = \frac{1}{\sin\theta}\partial_\theta\sin\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial_\lambda^2,$$

θ and λ being any system of co-latitude and longitude.

Given $\partial_t B^3$ it is well known (see, for example, Backus 1966) that (21) has exactly one solution ϕ satisfying (17). When ∂V is a sphere, simple techniques for solving (21) are reviewed at length by Backus (1958). If $\partial_t B^3$ is known, the irrotational part of $B^3\mathbf{v}$ is completely determined by (16).

We should note that if flux diffusion is negligible over the surface of the core except in a few small patches, then (21) will apply except in those patches. But then ϕ is no longer uniquely determined unless its boundary values at the edges of the patches are known. In the rest of the present paper we assume that (20) holds over all of ∂V.

No matter how ψ in (20) is chosen, the resulting $\mathbf{v}$ will satisfy (16), so (16) contains no information about the solenoidal part of $B^3\mathbf{v}$. However, we do get some further information about $\mathbf{v}$ and some information about ψ by demanding that $\mathbf{v}$ be continuous and examining the points where $B^3 = 0$.

A point on ∂V where $B^3 = 0$ will be called a null-flux point. We consider only the simplest case. We assume that the set of all null-flux points consists of a finite number of continuously differentiable, non-self-intersecting closed curves, to be called null-flux curves. We assume that no null-flux point lies on more than two null-flux curves, and we call a null-flux point ordinary or double according as it lies on one or two null-flux curves. To each point on a null-flux curve C we assign a unit vector $\boldsymbol{\nu}$ orthogonal to C and $\hat{\mathbf{n}}$ at that point and so as to be continuous on C. Then $\hat{\boldsymbol{\nu}}$ is tangent to ∂V everywhere on C. We assume that at a double null-flux point the two null-flux curves C_1 and C_2 meet at a non-zero angle, so that their normals $\hat{\boldsymbol{\nu}}_1$ and $\hat{\boldsymbol{\nu}}_2$ are linearly independent at the point of intersection. We assume that B^3 vanishes only to first order at ordinary null-flux points and only to second order at double null-flux points. That is, at every ordinary null-flux point $\hat{\boldsymbol{\nu}}.\nabla_S B^3 \neq 0$, and at a double null-flux point $\hat{\boldsymbol{\nu}}_1.\nabla_S\nabla_S B^3.\hat{\boldsymbol{\nu}}_2 \neq 0$. We assume B^3 twice continuously differentiable on ∂V, so that $\nabla_S B^3 = \mathbf{0}$ at a double null-flux point.

We call any maximal connected set of null-flux points a null-flux web.

At a null-flux point, (16) takes the form

$$\partial_t B^3 + \mathbf{v}.\nabla_S B^3 = 0. \tag{22}$$

At an ordinary null-flux point, $\hat{\mathbf{v}}.\nabla_S B^3 = (\hat{\boldsymbol{\nu}}.\nabla_S B^3)(\hat{\boldsymbol{\nu}}.\mathbf{v})$ so we can write (22) as

$$\hat{\boldsymbol{\nu}}.\mathbf{v} = -\partial_t B^3/(\hat{\boldsymbol{\nu}}.\nabla_S B^3). \tag{23}$$

At a double null-flux point we must have

$$\partial_t B^3 = 0 \tag{24}$$

if $\mathbf{v}$ is bounded.

From (20), at every null-flux point we must have

$$\nabla_S \psi = -\hat{\mathbf{n}} \times \nabla_S \phi \tag{25}$$

if $\mathbf{v}$ is bounded. Since ϕ is determined from $\partial_t B^3$ by (21), it follows that ψ is determined to within an additive constant on every null flux web. But will we really be able to solve (25) for a single valued ψ on each null-flux web? On ∂V let A be a patch whose boundary ∂A is a null-flux curve. If ψ is single-valued on ∂A we must have

$$\oint_{\partial A} \nabla_S \psi.(\hat{\boldsymbol{\nu}} \times \hat{\mathbf{n}})\,\mathrm{d}l = 0,$$

where l is arc length along ∂A. But from (25) this means

$$\oint_{\partial A} \nabla_S \phi.\hat{\boldsymbol{\nu}}\,\mathrm{d}l = 0,$$

whence Gauss's theorem implies $$\int_A \nabla_S^2 \phi\,\mathrm{d}A = 0.$$

Then it follows from (21) that $$\int_A \partial_t B^3\,\mathrm{d}A = 0. \tag{26}$$

Now we have two conditions which $\partial_t B^3$ must satisfy if it is to be produced by advection of B^3 in a continuous velocity field $\mathbf{v}$ on ∂V. At any double null-flux point we must have (24).

And if A is any patch on ∂V whose boundary ∂A is a null-flux curve we must have (26). If either of these conditions fails, no continuous $\mathbf{v}$ will produce $\partial_t B^3$ from B^3 by advection, and we must invoke flux diffusion.

We have shown that (24) and (26) are necessary conditions for the existence of a continuous $\mathbf{v}$ which satisfies (16). Now we show that those conditions are also sufficient. First we solve (21) for ϕ. If (26) holds then it will be possible to solve (25) for ψ on each null-flux web, to within an additive constant which we choose arbitrarily. On the part of ∂V where $B^3 \neq 0$, we choose ψ arbitrarily, making sure that it is twice continuously differentiable on all of ∂V and that $\hat{\boldsymbol{\nu}}.\nabla_S\psi = -(\hat{\boldsymbol{\nu}}\times\hat{\mathbf{n}}).\nabla_S\phi$ on each null-flux curve. In this fashion we construct a ψ which we will call ψ_0. Substituted in equation (20), it gives a continuous $\mathbf{v}_0$ which satisfies (16).

It remains to find all other 'eligible' velocity fields, that is, all other tangent velocity fields $\mathbf{v}$ on ∂V which satisfy (16) for the given functions $\partial_t B^3$ and B^3. Because (21) determines ϕ in (20), any eligible $\mathbf{v}$ must be such that $B^3\mathbf{v}$ and $B^3\mathbf{v}_0$ have the same irrotational part. Hence for any eligible $\mathbf{v}$ there is a scalar ψ_1 such that

$$B^3\mathbf{v} = B^3\mathbf{v}_0 - \hat{\mathbf{n}}\times\nabla_S\psi_1. \tag{27}$$

From (25), $\nabla_S\psi_1 = 0$ at every null-flux point, so ψ_1 is constant on every null-flux web. Conversely, if ψ_1 is any twice continuously differentiable function on ∂V which is constant on every null-flux web and has $\hat{\boldsymbol{\nu}}.\nabla_S\psi_1 = 0$ on every null-flux curve, then (27) gives an eligible $\mathbf{v}$, a continuous $\mathbf{v}$ which satisfies (16).

The physical meaning of the conditions on $\partial_t B^3$ and $\mathbf{v}$ at null-flux curves is simple. We can 'see' B^3 and $\partial_t B^3$ but not $\mathbf{v}$. A null-flux curve always consists of the same fluid particles (other level lines of B^3 do not), so we can follow the motion of such a curve of particles, but we cannot see how the individual particles slip along the curve. On any null-flux curve we can see how fast the fluid moves normal to the curve, whence (23). At a double null-flux point we have the components of $\mathbf{v}$ in two different directions, the normals to the two moving null-flux curves, so we know $\mathbf{v}$ unambiguously at every double null-flux point.

The physical meaning of (26) is equally simple. If the patch A is bounded by a null-flux curve, it always consists of the same fluid particles, so the magnetic flux through it is constant. But if $\hat{\boldsymbol{\nu}}$ points out of A then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A B^3\,\mathrm{d}A = \int_A \partial_t B^3\,\mathrm{d}A + \int_{\partial A} B^3\,\hat{\boldsymbol{\nu}}.\mathbf{v}\,\mathrm{d}l.$$

The vanishing of the left-hand side and of B^3 on ∂A gives (26).

Using Cain's (1967) harmonic amplitudes Booker (1968) has looked at B^3 and $\partial_t B^3$ in on the core/mantle boundary. Within the limits of resolution permitted by the absence of spherical harmonics above angular order $l = 10$, Booker finds three null-flux curves and no double null-flux points. He finds that (26) is not satisfied unless one excludes the dipole from $\partial_t B^3$ or admits in $\partial_t B^3$ spherical harmonics whose amplitude is uncertain by 100 per cent. It will be interesting to see whether this situation persists as the harmonic amplitudes are improved.

7. Velocity information from $\hat{\mathbf{n}}\cdot\mathbf{B}$ and $\hat{\mathbf{n}}\times\mathbf{B}$

For completeness we consider the possibility that B^i in (15) are known just inside ∂V. We assume that we have already extracted from (14) all its information about $\mathbf{v}$, so that ϕ is known on ∂V and $\hat{\mathbf{v}}\cdot\mathbf{v}$ is known on every null-flux curve. In the Earth's core we can assume $\nabla\cdot\mathbf{v} = 0$, but for completeness we also consider the possibility that $\nabla\cdot\mathbf{v}$ is unknown.

No matter how we choose $\mathbf{v}$ to satisfy (14) and no matter what we assume about $\nabla\cdot\mathbf{v}$, we can always solve (15) for $\partial_3 v^i$ at any point on ∂V where $B^3 \neq 0$. Therefore (15) gives information about $\mathbf{v}$ on ∂V only at points where $B^3 = 0$, that is, on null-flux curves. To examine a particular null-flux curve C, we introduce orthogonal coordinates x^1, x^2 on ∂V such that the equation of C is $x^2 = 0$. We define

$$g = (g_{11}g_{22})^{\frac{1}{2}},$$

$$\gamma = (g_{11}/g_{22})^{\frac{1}{2}},$$

so that (20) can be written

$$gB^3v^1 = \gamma^{-1}\partial_1\phi + \partial_2\psi,$$

$$gB^3v^2 = \gamma\,\partial_2\phi - \partial_1\psi.$$

Then $gB^3\partial_2 v^i + v^i\partial_2(gB^3) = w^i$, where

$$w^1 = \partial_2(\gamma^{-1}\partial_1\phi + \partial_2\psi),$$

$$w^2 = \partial_2(\gamma\,\partial_2\phi - \partial_1\psi).$$

From these definitions

$$\partial_1 w^1 + \partial_2 w^2 = \partial_2[\partial_1(\gamma^{-1}\partial_1\phi) + \partial_2(\gamma\,\partial_2\phi)] = \partial_2[g\nabla_S^2\phi],$$

so from (21)

$$\partial_1 w^1 + \partial_2 w^2 + \partial_t\partial_2(gB^3) = 0. \tag{28}$$

Moreover

$$(gB^3)\,\partial_2^2 v^i + 2\,\partial_2(gB^3)\,\partial_2 v^i + v^i\,\partial_2^2(gB^3) = \partial_2 w^i,$$

so on the null-flux curve $x^2 = 0$ we have

$$\left.\begin{aligned} v^i &= \frac{w^i}{\partial_2(gB^3)},\\ \partial_2 v^i &= \tfrac{1}{2}\partial_2\left[\frac{w^i}{\partial_2(gB^3)}\right].\end{aligned}\right\} \tag{29}$$

If we multiply the second of equations (15) by $2B^2\,\partial_2(gB^3)$ and make use of (28) and (29) we obtain the following equation, valid on the null-flux curve $x^2 = 0$

$$\partial_1[w^1(B^2)^2] = 2B^1B^2(\partial_1 v^2)\,\partial_2(gB^3) - (\partial_t + v^2\partial_2)[(B^2)^2\,\partial_2(gB^3)] - 2(B^2)^2(\nabla\cdot\mathbf{v})\,\partial_2(gB^3). \tag{30}$$

The first of equations (15) is

$$B^2\,\partial_2 v^1 = (\nabla\cdot\mathbf{v})\,B^1 + \partial_t B^1 + v^1\partial_1 B^1 + v^2\,\partial_2 B^1 - B^1\,\partial_1 v^1. \tag{31}$$

From (30) and (31) it is clear that the points on the null-flux curve C where $B^2 = 0$ will play an important role. We call such points 'touch points' because they are points of tangency of $\mathbf{B}$ and C. We must consider two cases:

Case 1. $\nabla\cdot\mathbf{v}$ *unknown*. Then we can always choose $\nabla\cdot\mathbf{v}$ on C (the null-flux curve $x^2 = 0$) so that (30) can be integrated around C to give a single-valued, continuous v^1

while (31) can be solved for a continuous $\partial_2 v^1$. Therefore equations (15) give no new information about $\mathbf{v}$ on ∂V, and no new conditions on $\mathbf{B}$ and $\partial_t \mathbf{B}$.

Case 2*A*. $\nabla . \mathbf{v} = 0$ *and C has no touch points*. Then if w^1 is to be single-valued on the closed curve C we must have

$$\oint_C H \mathrm{d}x^1 = 0 \tag{32}$$

where H is the right-hand side of (30). If this necessary condition is satisfied, we can solve (30) for $w^1(B^2)^2$ to within an arbitrary additive constant. Having chosen this constant we have v^1 on C. Then we can solve (31) for $\partial_2 v^1$ on C. From the definitions of w^i it follows that now on C we know ψ, $\partial_2\psi$, $\partial_2^2\psi$ and $\partial_2^3\psi$. We can choose ψ arbitrarily where $B^3 \neq 0$, as long as we fit it with three continuous derivatives onto the null-flux curve.

Case 2*B*. $\nabla . \mathbf{v} = 0$ *and C has at least one touch point*. Then if x_a^1 and x_b^1 are the x^1 coordinates of any two touch points on C, or the coordinates of a single touch point before and after one traversal of C, we must have

$$\int_{x_a^1}^{x_b^1} H \mathrm{d}x^1 = 0 \tag{33}$$

where H is again the right-hand side of (30). If (33) is satisfied for all touch points on C then we can integrate (30) for $w^1(B^2)^2$ and now $w^1(B^2)^2$ is uniquely determined on C; only one choice of the additive constant of integration makes w^1 continuous at the touch points. The value for $\partial_2 v^1$ obtained from (31) will be continuous at the touch points if and only if at every touch point we have

$$\partial_t B^1 + v^1 \partial_1 B^1 + v^2 \partial_2 B^1 - B^1 \partial_1 v^1 = 0. \tag{34}$$

Since v^1 as well as v^2 has been determined on C by this stage of the discussion, (34) is another condition on $\partial_t \mathbf{B}$ and $\mathbf{B}$, failure of which means that no $\mathbf{v}$ continuous on ∂V can satisfy (16).

We can summarize the results of case 2, the case of interest for the Earth's core: if $\hat{\mathbf{n}} \times \mathbf{B}$ is observable just below ∂V, we have further conditions on $\mathbf{B}$ and $\partial_t \mathbf{B}$ on null-flux curves, necessary for (9) to have any continuous solution $\mathbf{v}$ on ∂V. If these conditions ((32) on null-flux curves without touch points and (33) and (34) on null-flux curves with touch points) are satisfied, then we have information about $\partial_2^2\psi$ and $\partial_2^3\psi$ on null-flux curves, but ψ in (20) can be chosen arbitrarily where $B^3 \neq 0$ as long as ψ and its derivatives of orders one, two and three fit continuously into the values determined on the null-flux curves.

The physical significance of the conditions on $\mathbf{v}$, $\partial_t \mathbf{B}$ and $\mathbf{B}$ obtained from (15) is less clear than for (14). If a fluid particle is a touch point at one instant, it is always so, and its motion can be followed because the touch point is always identifiable. The velocity of a touch point is obtainable from $\mathbf{B}$ and $\partial_t \mathbf{B}$ whether $\nabla . \mathbf{v}$ vanishes or not. The integral conditions (32) and (33) are perhaps a description of flux constancy through a fluid ribbon one of whose edges is C, but I have not succeeded in demonstrating this conjecture.

8. Velocity information obtainable from $\hat{\mathbf{n}} \times \mathbf{E}$ and $\hat{\mathbf{n}} . \mathbf{B}$

Let $\mathbf{E}$ denote the electric field vector. If we know $\hat{\mathbf{n}} \times \mathbf{E}$ and $\hat{\mathbf{n}} . \mathbf{B}$ just outside the core, we can find $\mathbf{v}$ at the top of the free stream very easily. In a perfect conductor we have

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}.$$

On the boundary ∂V, if we take the cross product of $\hat{\mathbf{n}}$ with the foregoing equation we obtain

$$\hat{\mathbf{n}} \times \mathbf{E} = (\hat{\mathbf{n}}.\mathbf{v})\, \mathbf{B} - (\hat{\mathbf{n}}.\mathbf{B})\, \mathbf{v}.$$

Since $\hat{\mathbf{n}}.\mathbf{v} = 0$, we have immediately

$$\hat{\mathbf{n}} \times \mathbf{E} = -(\hat{\mathbf{n}}.\mathbf{B})\, \mathbf{v}.$$

Maxwell's equations imply that at the boundary ∂V of a perfect conductor both $\hat{\mathbf{n}} \times \mathbf{E}$ and $\hat{\mathbf{n}}.\mathbf{B}$ are continuous. Therefore if we can measure these quantities just outside ∂V we can determine $\mathbf{v}$ uniquely everywhere on ∂V. The resulting $\mathbf{v}$ will be continuous if and only if $\hat{\mathbf{n}} \times \mathbf{E}$ vanishes at least linearly on the null-flux curves.

Unfortunately there are serious obstacles to measuring $\mathbf{E}$ at the core-mantle boundary. Let $\mathbf{A}$ be a magnetic vector potential outside the core, so that $\mathbf{B} = \nabla \times \mathbf{A}$, $\nabla . \mathbf{A} = 0$, and $\mathbf{E} = -\partial_t \mathbf{A} - \nabla W$. Ohm's law in the mantle is $\nabla \times \mathbf{B} = \mu_0 \sigma \mathbf{E}$ if we neglect displacement currents. Therefore the equation governing the potential W is

$$\nabla^2 W + \nabla W . \nabla \ln \sigma = -\partial_t \mathbf{A} . \nabla \ln \sigma.$$

The effect of a non-vanishing σ in the mantle does not disappear at low frequencies for the electric field as it does for the magnetic field. Therefore the large uncertainty in σ in the mantle probably makes it premature to attempt to extrapolate to the core an $\mathbf{E}$ observed at the top of the mantle. This is true even if $\mathbf{E}$ is measured on the ocean bottoms to eliminate the very large effect of the conductivity of sea water.

9. Westward drift

From what has been said above, it is clear that there is no unique solution to the problem which Kahle *et al.* (1967) have set themselves, to find from $\partial_t \mathbf{B}$ and $\mathbf{B}$ on ∂V that $\mathbf{v}$ on ∂V which best satisfies (16) in the least-squares sense. They find a unique solution only because they have truncated the vector spherical harmonic expansion of $\mathbf{v}$. But as we saw in §6, without that truncation the non-uniqueness of $\mathbf{v}$ is not simply a question of lack of small-scale resolution. Two different $\mathbf{v}$'s can satisfy (16) and yet differ by an amount comparable to their own amplitudes on a length scale equal to the circumference of the core.

Despite this indeterminancy, $\mathbf{B}$ and $\partial_t \mathbf{B}$ can be used to answer questions about $\mathbf{v}$ if those questions are posed in a way which explicitly recognizes the non-uniqueness of $\mathbf{v}$. As an example we discuss the question of westward drift at the top of the free stream in the core. We ignore (15) because even if $\hat{\mathbf{n}} \times \mathbf{B}$ is known just below ∂V the information which (15) gives about $\mathbf{v}$ is confined to a few points and curves. With the limited spatial resolution now available in $\partial_t \mathbf{B}$ at the core, such information is much less reliable than areawide information of the sort conveyed by (14).

We let $\partial_t B^3$ and B^3 denote the values observed just above ∂V. We call a continuous velocity field on ∂V which is tangent to ∂V and with B^3 and $\partial_t B^3$ satisfies (16) (or equivalently (14)) an eligible velocity field. We suppose that (24) is satisfied at every double null-flux point and (26) is satisfied for every patch A bounded by a null-flux curve, so that B^3 and $\partial_t B^3$ admit at least one eligible velocity field $\mathbf{v}_0$. Then the other eligible $\mathbf{v}$ are precisely those of the form

$$\mathbf{v} = \mathbf{v}_0 - (B^3)^{-1}\, \hat{\mathbf{n}} \times \nabla_S \psi, \tag{35}$$

where ψ is in $\mathscr{E}$, the class of all functions which are constant on each null flux web and have $\hat{\mathbf{v}}.\nabla_S\psi = 0$ on each null-flux curve.

Given the observed $\partial_t B^3$ and B^3 any of the velocity fields (35) is eligible, so we cannot ask whether there is westward drift of core fluid, or latitude dependence of the westward drift of the core fluid, at the top of the free stream, unless we have other data or other conditions on $\mathbf{v}$ which are not now available. (That the magnetic field itself shows a westward drift was, of course, clearly established by Bullard, Freedman, Gellman & Nixon 1950.) We can ask, however, what is the eligible velocity field which in the least-squares sense is most nearly a uniform rigid rotation about the geographic polar axis of the Earth. This velocity field has no particular claim to reality, but if its r.m.s. deviation from pure rigid rotation is much less than its r.m.s. value we may be impelled to examine core mechanisms capable of producing such a result.

To find the velocity field in question, we seek ψ in $\mathscr{E}$ and that constant Ω which minimize the integral

$$2Q = \int_{\partial V} [\mathbf{v}_0 - (\hat{\mathbf{n}}\times\nabla_S\psi)/B^3 - \hat{\boldsymbol{\lambda}}\Omega R\sin\theta]^2\,\mathrm{d}A,$$

where $\hat{\boldsymbol{\lambda}}$ is the unit vector in the direction of increasing geographic longitude λ, θ is geographic colatitude, and R is the radius of the spherical core. A positive Ω is an eastward drift of the core relative to the mantle.

The integral can be written

$$2Q = \int_{\partial V} [\nabla_S\psi/B^3 + \hat{\mathbf{n}}\times\mathbf{v}_0 + \hat{\boldsymbol{\theta}}\Omega R\sin\theta]^2\,\mathrm{d}A. \tag{36}$$

For given ψ and Ω, small variations $\delta\psi$ and $\delta\Omega$ change Q by an amount δQ which, after an integration by parts and a use of $\delta\psi$'s membership in $\mathscr{E}$, can be written

$$\delta Q = R\delta\Omega\int_{\partial V}\sin\theta[\partial_\theta\psi/(RB^3) - \hat{\boldsymbol{\lambda}}.\mathbf{v}_0 + \Omega R\sin\theta]\,\mathrm{d}A$$
$$-\int_{\partial V}\delta\psi\nabla_S.[(B^3)^{-2}\nabla_S\psi + (B^3)^{-1}(\hat{\mathbf{n}}\times\mathbf{v}_0 + \hat{\boldsymbol{\theta}}\Omega R\sin\theta)]\,\mathrm{d}A. \tag{37}$$

If ψ and Ω in fact minimize (36) then $\delta Q = 0$ for any $\delta\Omega$ and any $\delta\psi$ in $\mathscr{E}$. Hence the minimizing Ω and ψ satisfy

$$\Omega = \frac{3}{8\pi R}\int_{\partial V}\sin\theta[\hat{\boldsymbol{\lambda}}.\mathbf{v}_0 - \partial_\theta\psi/(RB^3)]\,\mathrm{d}A, \tag{38}$$

$$\nabla_S^2\psi - 2(B^3)^{-1}\nabla_S B^3.\nabla_S\psi = B^3[\nabla_S.\hat{\mathbf{n}}\times\mathbf{v}_0 + 2\Omega R\cos\theta]$$
$$-\nabla_S B^3.[\hat{\mathbf{n}}\times\mathbf{v}_0 + \hat{\boldsymbol{\theta}}\Omega R\sin\theta]. \tag{39}$$

The singularity of the elliptic equation (39) for ψ on the null-flux curves makes solution of that equation more than a routine question. For the corresponding plane problem when there is only a single, straight null-flux line Schechter (1960) has proved existence and uniqueness for the Dirichlet problem. We will assume that Schechter's result generalizes to the present case, but we must emphasize that we have not proved this assumption. We assume that if $W_1, \ldots, W_n$ are the null-flux webs on ∂V and f is any suitably smooth function on ∂V then

$$\nabla_S^2\psi - 2(B^3)^{-1}\nabla_S B^3.\nabla_S\psi = f \tag{40}$$

has exactly one solution ψ in $\mathscr{E}$ which takes any specified values $K_1, \ldots, K_n$ on the webs $W_1, \ldots, W_n$. (Schechter's result would permit K_i to be a function of position on W_i; we need only the case where K_i is constant.) The solution which is 1 on W_i and zero on the other webs when in (40)

$$f = B^3\nabla_S . \hat{\mathbf{n}} \times \mathbf{v}_0 - \nabla_S B^3 . \hat{\mathbf{n}} \times \mathbf{v}_0$$

we denote by $\tilde{\psi}_i$. The solution which vanishes on all the webs when

$$f = R(2\cos\theta - \sin\theta\,\partial_\theta)\,B^3$$

we call $\tilde{\psi}_0$. Then the solution of (39) which takes the value K_i on W_i is

$$\psi = \Omega\tilde{\psi}_0 + \sum_{i=1}^{n} K_i \tilde{\psi}_i.$$

To simplify the notation we relabel Ω above and in (36) and (39) as K_0. Then

$$\psi = \sum_{i=0}^{n} K_i \tilde{\psi}_i. \tag{41}$$

Now $\tilde{\psi}_0, \tilde{\psi}_1, \ldots, \tilde{\psi}_n$ can all be calculated from $\partial_t B^3$, B^3, and $\mathbf{v}_0$ if as we assume, Schechter's theorem generalizes to the present geometry. We don't yet know the constants $K_0, \ldots, K_n$; they must be chosen to minimize Q when (41) is substituted in (36). This substitution gives Q as an inhomogeneous polynomial of second degree in $K_0, \ldots, K_n$ whose homogeneous quadratic part is positive definite. Thus there is a unique choice of $K_0, K_1, \ldots, K_n$ which minimizes Q when ψ is given by (41). The minimizing K's are found as the solution of $n+1$ linear inhomogeneous equations, one of which is (38). The system is non-singular because the matrix of the homogeneous part is symmetric and positive definite.

Essentially the same technique can be used to investigate latitude-dependent westward drift if we are willing to consider latitude dependent $\Omega(\theta)$ with only finitely many degrees of freedom, such as

$$\Omega(\theta) = \sum_{j=0}^{m} \Omega_j P_j(\cos\theta), \tag{42}$$

where $\Omega_0, \Omega_1, \ldots, \Omega_m$ are constants and $P_j(x)$ is the jth Legendre polynomial. We seek that eligible velocity $\mathbf{v}$ which in the least squares sense is most nearly of the form $\hat{\boldsymbol{\lambda}}R\Omega(\theta)$ for some choice of $\Omega_0, \ldots, \Omega_m$ in (42). That is, we seek the ψ in $\mathscr{E}$ and the constants Ω_0, $\Omega_1, \ldots, \Omega_m$ which minimize (36) when in (36) Ω has the form (42). The discussion proceeds exactly as before. The minimizing ψ has the form

$$\psi = \sum_{i=0}^{m+n+1} K_i \tilde{\psi}_i, \tag{43}$$

where the $\tilde{\psi}_i$ are obtained from $\partial_t B^3$, B^3 and $\mathbf{v}^0$ and the constants K_i are then determined by substituting (43) in (36) and minimizing Q as a function of $K_0, K_1, \ldots, K_{m+n+1}$. The constants $K_{m+1}, \ldots, K_{m+n+1}$ are the values of ψ on the null-flux webs $W_1, \ldots, W_n$, while $K_0, K_1, \ldots, K_m$ are the $\Omega_0, \Omega_1, \ldots, \Omega_m$ in (43). The K_i emerge as the solutions of $m+n+1$ linear inhomogeneous equations, the matrix of whose homogeneous part is symmetric and positive-definite and hence nonsingular.

If we try to generalize the foregoing procedure to the case that $\Omega(\theta)$ is an arbitrary function of θ we encounter serious complications. We must find the Green function for

(40) with $\psi = 0$ on the null-flux webs. Then from (39) and this Green function we must express ψ as a function of $\Omega(\theta)$. We must substitute the result in (36) and minimize Q as a functional of Ω. The result is a singular Fredholm equation of the first kind for $\Omega(\theta)$. Alternatively, we can use (39) and the analogue of (38) obtained from (37) when $\delta\Omega$ and Ω are functions of θ, namely

$$\Omega(\theta) = \frac{1}{2\pi R \sin\theta} \int_0^{2\pi} [\hat{\boldsymbol{\lambda}} \cdot \mathbf{v}_0 - \partial_\theta \psi/(RB^3)]\, \mathrm{d}\lambda. \tag{44}$$

Inserting (44) in (39) gives an integro-differential equation for ψ. Neither this nor the Green's function approach makes it clear whether the mathematical problem is well posed when Ω is an arbitrary function of θ. It would appear, however, that representations in the form (41) should suffice for all practical purposes.

If ψ is given, whether it minimizes (36) or not, the Ω of (38) describes the rigid rotation which best fits the velocity (35) in the least-squares sense. A. Toomre (1960, private communication), Hide (1966) and Malkus (1967) have suggested that perhaps Ω is zero and the westward drift of the magnetic anomalies is the surface manifestation of a tendency for Alfvén-inertial waves in the core to have predominantly westward phase-velocities. From (38) it is clear that we cannot decide between this wave hypothesis and the suggestion of Bullard *et al.* (1950) that there is a net westward drift of fluid in the upper core. We can always find a ψ in $\mathscr{E}$ which gives Ω in (38) any value whatever.

Although we cannot decide between the two hypotheses by examining only the secular variation, we can at any rate sharpen an observation recently made by Stewartson (1967, p. 183) who says of westbound hydromagnetic waves in the core as a cause of the westward drift in the secular variation, '...the oscillations cannot be observed directly outside the shell under the assumed conditions and can only be manifested indirectly'. The test of whether a scalar field $f(\theta, \lambda, t)$ on ∂V tends to drift westward is a comparison of $\partial_t f$ with $\partial_\lambda f$. In the simplest case, $\partial_t f/\partial_\lambda f$ is constant. A more sophisticated test for the presence of appreciable westward drift in f is the demand that there exists a constant Ω or a function $\Omega(\theta)$ such that the ratio

$$\int_{\partial V} |\partial_t f - \Omega \partial_\lambda f|^2\, \mathrm{d}A \Big/ \int_{\partial V} |\partial_t f|^2\, \mathrm{d}A$$

is considerably less than 1. A still more sophisticated test is to examine the Fourier components of f (Hide 1966). The test for whether a tangent vector field $\mathbf{v}(\theta, \lambda, t)$ drifts west on ∂V is to test the two fields $\hat{\boldsymbol{\theta}} \cdot \mathbf{v}(\theta, \lambda\, t)$ and $\hat{\boldsymbol{\lambda}} \cdot \mathbf{v}(\theta, \lambda, t)$ as if they were scalars, and to require that their westward drifts, if present, be the same. In any case, the test for westward drift involves a comparison of time derivatives with derivatives in longitude.

Now we assume, with the proponents of the wave theory of the westward drift, that in discussing the secular variation we can treat the free stream in the core as if it were a perfect conductor. Then Alfvén-inertial waves can certainly exist in the core, but the magnetic field which supports them is the toroidal field. The poloidal field B^3 at the surface of the core is relatively much smaller and almost without dynamical effect. It acts only as a tracer of the fluid motion in that it is governed by (16). But in (16) $\partial_t \mathbf{v}$ does not appear. In other words, the behaviour of $\mathbf{v}$ with time and the question of whether the pattern of $\mathbf{v}$ shows a westward drift are completely irrelevant to the equation by which the free-stream motion determines the secular variation. In so far as $\mathbf{v}$ governs $\partial_t B^3$ as if the free stream were a perfect conductor, any westward drift in the pattern of **v** is irrelevant to the observed westward drift in the pattern of B^3.

An analogy may clarify the situation. Consider an observer in a windowless, 24 h Earth satellite at the longitude of the Rocky Mountains. Suppose advanced electronic surveying techniques enable him to see on a television screen an instantaneous topographic map of the western United States. A Love wave travelling from east to west across the Rockies will not produce westbound waves on the map. It will produce heterodyning between the Love wave-numbers and the wave-numbers in the topography.

Appendix I. Solving the secular equation

We want to learn for which values of α and β or γ and β all the roots of (4) have

$$|\mathscr{R}m_\tau| > u.$$

From (5) we infer that to deal with periods in the Earth's core shorter than a century it probably suffices to consider $u \leqslant 0{\cdot}3$ unless the electrical conductivity of the core is as low as 10^5 mho/m, in which case we should consider $u \leqslant 0{\cdot}5$.

It will be useful to note the solution of (4) in two special cases. First, when $|\beta_\tau| \ll 1$ and α_τ is finite,

$$m_{\kappa\tau} = e^{-\frac{1}{4}i\pi} + O(|\beta_\tau|(\alpha_\tau^2+1)^{\frac{1}{2}}),$$

$$m_{\nu\tau} = |\beta_\tau|^{\frac{1}{2}} e^{\frac{1}{4}i\pi \operatorname{sgn} \beta_\tau} + O(|\beta_\tau|^{\frac{3}{2}}(\alpha_\tau^2+1)^{\frac{1}{2}}).$$

Thus for any positive u, the axis $\beta_\tau = 0$ lies outside the region where all roots of (4) satisfy $|\mathscr{R}m_\tau| > u$. Secondly, when $|\beta_\tau| \gg 1$ and γ is finite, so $\alpha_\tau \ll 1$, then

$$\left.\begin{aligned} m_{\kappa\tau} &= e^{-\frac{1}{4}i\pi} + O(|\beta_\tau|^{-1}(\gamma^2+1)^{\frac{1}{2}}),\\ m_{\nu\tau} &= |\beta_\tau|^{\frac{1}{2}} e^{-\frac{1}{4}i\pi \operatorname{sgn} \beta_\tau} + O(|\beta_\tau|^{-\frac{1}{2}}(\gamma^2+1)^{\frac{1}{2}}).\end{aligned}\right\} \tag{45}$$

Therefore if $u^2 < \frac{1}{2}$ both ends of the axis $\alpha = 0$, that is $\beta_\tau \gg 1$ and $\beta_\tau \ll -1$, lie inside the region where all roots of (4) satisfy $|\mathscr{R}m_\tau| > u$.

It remains to find the dividing line between the two regions, i.e. the level curve

$$\mathscr{R}m_\tau(\alpha_\tau, \beta_\tau) = u.$$

For a given value of u, we seek the real α_τ and β_τ or β_τ and γ which produce roots of (4) in the form

$$m_\tau = u(1+ix), \tag{46}$$

where x is real. To find these values of α_τ, β_τ and γ we substitute (45) in (4), equate real and imaginary parts to zero, and solve the resulting pair of simultaneous equations for α and β. The result can be written

$$\alpha_\tau = |a|/u^2,$$

$$\beta_\tau = 2u^2 b,$$

where

$$a = \frac{1-x^2}{(1+x^2)^2}\left[\frac{1+4u^2x+u^4(1+x^2)^2}{1+2u^2x}\right], \tag{47}$$

$$b = \frac{(1+x^2)^2(1+2u^2x)}{2[2x+u^2(1+x^2)^2]}. \tag{48}$$

In addition

$$\gamma = \frac{(1-x^2)[1+4u^2x+u^4(1+x^2)^2]}{2x+u^2(1+x^2)^2}. \tag{49}$$

To study these expressions we use the following fact: if $P(u, x)$ is a real polynomial in x depending on a real parameter u, and if $\partial_u P$ is continuous, then as u varies the number of roots x of $P(u, x) = 0$ can change only at values of u and x which satisfy not only $P(u, x) = 0$ but also $\partial_x P(u, x) = 0$.

From this principle (or by direct calculation) it follows that in (49) the polynomial $1+4u^2x+u^4(1+x^2)^2$ is positive for all real u and x except $u^2 = \frac{1}{2}$, $x = -1$. It also follows that the denominator in (49) is positive for all real x if $u^4 > \frac{27}{64}$, while if $u^4 = \frac{27}{64}$ there is a double zero at $x = -3^{-\frac{1}{2}}$, and if $0 < u^4 < \frac{27}{64}$ there are two real zeros, $x_1(u) < -3^{-\frac{1}{2}}$ and $x_2(u) > -3^{-\frac{1}{2}}$. In fact if $0 < u^4 < \frac{27}{64}$ we have

$$-1/2u^2 < x_1(u) < -3^{-\frac{1}{2}} < x_2(u) < 0$$

except that when $u^2 = \frac{1}{2}$ then $-(2u^2)^{-1} = x_1(u) = -1$. If u is very much less than 1,

$$x_1(u) = -(2/u^2)^{\frac{1}{3}}+O(u^{\frac{2}{3}}), \tag{50}$$

$$x_2(u) = -\tfrac{1}{2}u^2+O(u^6). \tag{51}$$

From its definition γ must be positive. Therefore when $0 < u^2 < \frac{1}{2}$ it is clear from (49) that the only possible values of x are in the intervals

$$x_1(u) < x < -1 \tag{52}$$

and
$$x_2(u) < x < 1. \tag{53}$$

Therefore the level line $\mathscr{R}m_\tau(\alpha_\tau, \beta_\tau) = u$ has two branches. On the first branch we have (52), $b(x) < 0$, and $b(-1) = -1$, while $b(x) \to -\infty$ as $x \to x_1(u)$. Moreover, $a(x) < 0$, $a(-1) = 0$, and

$$a(x_1(u)) = \frac{1-x_1(u)^2}{[1+x_1(u)^2]^2} = -\frac{u^2}{2}\left[\frac{1}{x_1(u)}-x_1(u)\right]. \tag{54}$$

On the second branch we have (53), $b(x) > 0$, and $b(1) = 1$, while $b(x) \to +\infty$ as $x \to x_2(u)$ from above. Moreover, $a(x) > 0$ and $a(1) = 0$ while

$$a(x_2(u)) = \frac{1-x_2(u)^2}{[1+x_2(u)^2]^2} = -\frac{u^2}{2}\left[\frac{1}{x_2(u)}-x_2(u)\right]. \tag{55}$$

Since $\alpha = |a|/u^2$, we are only concerned with $|a|$, not the sign of a. Level curves of $\mathscr{R}m_\tau = u$ are given in the b, $|a|$ plane for two values of u in figure 2. The first branch is on the left, the second branch on the right. The region where $|\mathscr{R}m_\tau| > u$ is the region below the level curves and above the b axis.

We recall now that $\beta_\tau = \kappa(\tau f-\omega)/\nu\omega$ so that, for the solutions of (2) whose time dependence is periodic with period $2\pi/|\omega|$, β can have either sign. Therefore we have two sets of level curves in the b, $|a|$ plane, one obtained from the other by reflexion in the $|a|$ axis. In order to have $|\mathscr{R}m_\tau| > u$ for all roots of (4) with both choices of τ and both signs of ω we must choose the point $(b, |a|)$ to lie below the lower of the two level curves.

When $u^4 < \frac{27}{64}$ the lower curve is always the first branch, so we ignore the second branch in the rest of this appendix. In figure 3 we show the first branch for various values of u. Here the horizontal axis is $|b|^{-1}$ so that the whole curve can be shown on one figure. The region where $|\mathscr{R}m_\tau| > u$ for all four roots of (4) with both choices of τ and both signs of ω is the bounded region in figure 3 enclosed by the axes of $|b|^{-1}$ and $|a|$ and the level curve labelled u.

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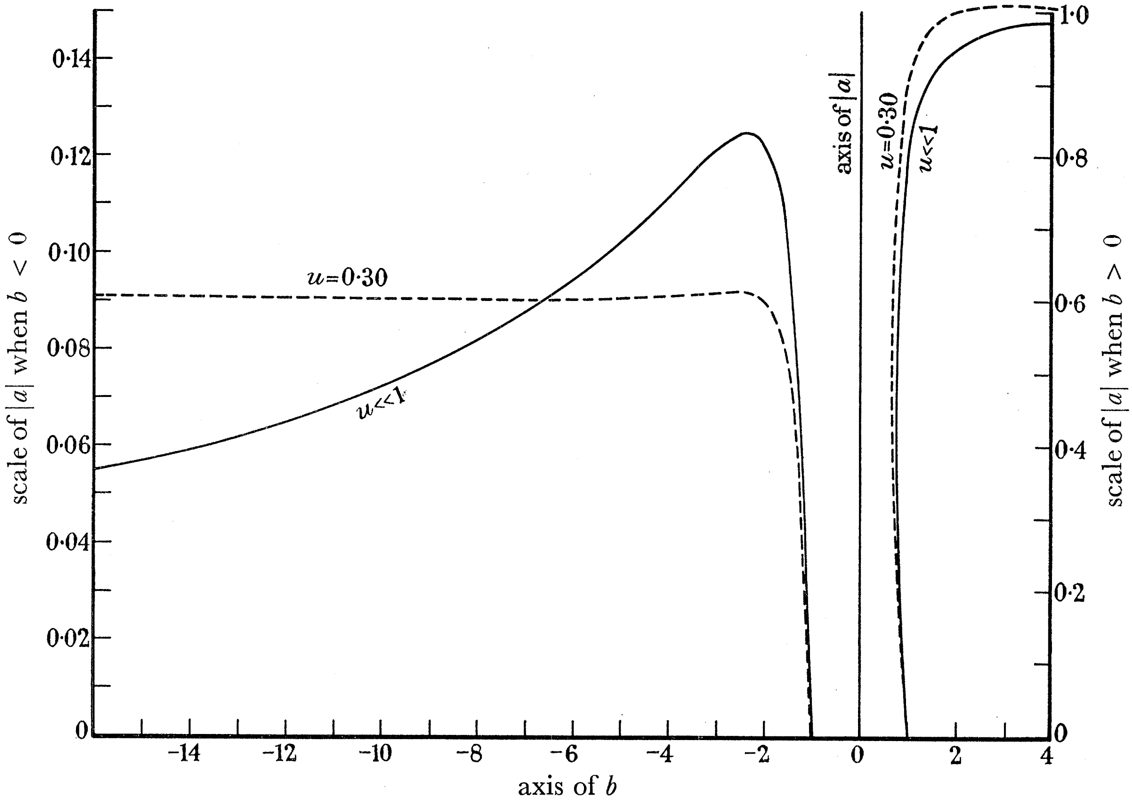

FIGURE 2. Level lines of $\mathscr{R}m_\tau\ (\alpha_\tau, \beta_\tau) = u$ in equation (4) for $u = 0{\cdot}3$ and $u \ll 1$. The vertical axis is $|a|$, the horizontal axis is b, and $\alpha = |a|u^{-2}$, $\beta = 2u^2b$. Note the different scales for $|a|$ when b is positive or negative.

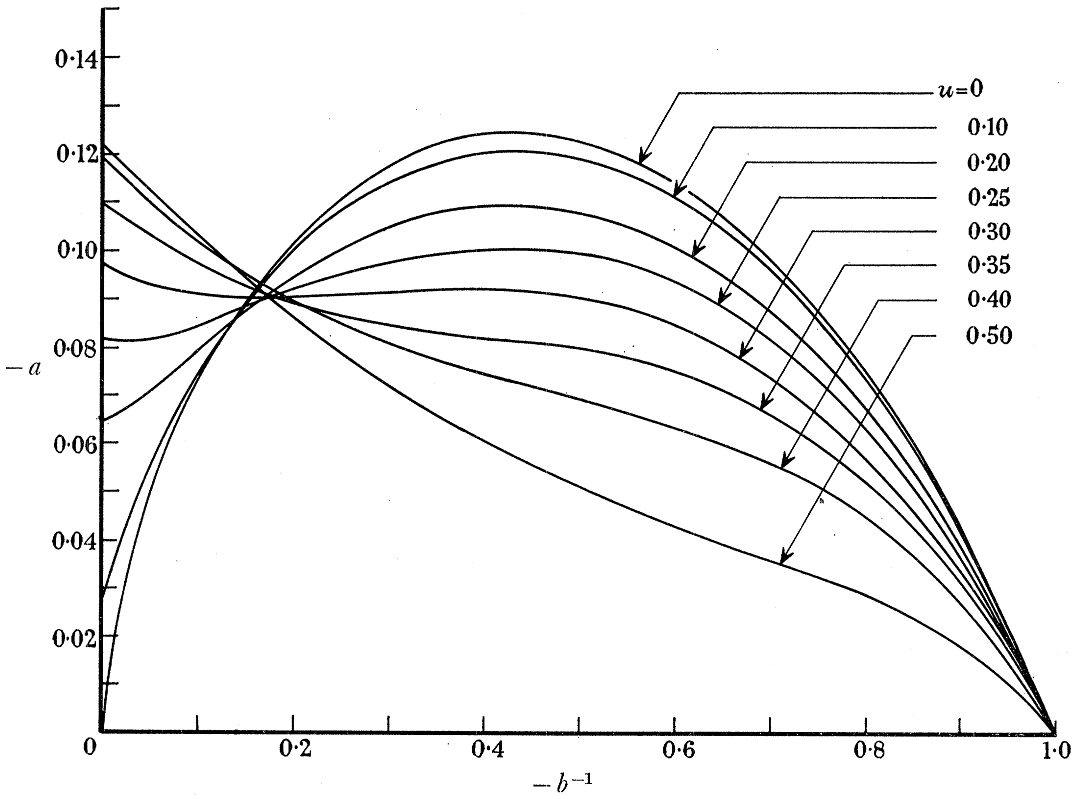

FIGURE 3. The most restrictive level line of $\mathscr{R}m_\tau(\alpha_\tau, \beta_\tau) = u$ (that with $\beta_\tau < 0$) plotted for various values of u less than 0·5. The vertical axis is $|a|$, the horizontal axis is $|b|^{-1}$, and $\alpha = |a|u^{-2}$, $\beta = 2u^2b$. If $(|b|^{-1}, |a|)$ lies below the curve labelled u then all roots of equation (4) satisfy $|\mathscr{R}m_\tau| > u$ for both signs of τ and ω. Note the local minimum of $|a|$ in the interval $0 < |b|^{-1} < 0{\cdot}1$, which appears at $u = 0{\cdot}2369$ and disappears at $u = 0{\cdot}3142$.

As u increases from zero, the level curves in figure 3 develop a local minimum near $|b|^{-1} = 0{\cdot}1$. This minimum first appears at $u = 0{\cdot}2369$, and it disappears at $u = 0{\cdot}3142$. The level line for a given u intersects the $|a|$ axis at $|a| = |a(x_2(u))|$. Figure 4 gives a graph of

$$g(u) = (2/u^2)^{\frac{2}{3}}\,|a(x_2(u))|$$

as a function of u in the interval $0 < u < 0{\cdot}5$. The small spur projecting down from this graph in the interval $0{\cdot}2369 < u < 0{\cdot}30$ is a graph of

$$h(u) = (2/u^2)^{\frac{2}{3}} \min_x |a(x, u)|,$$

where $\min_x |a(x, u)|$ is the local minimum shown in figure 3 near $|b|^{-1} = 0{\cdot}1$.

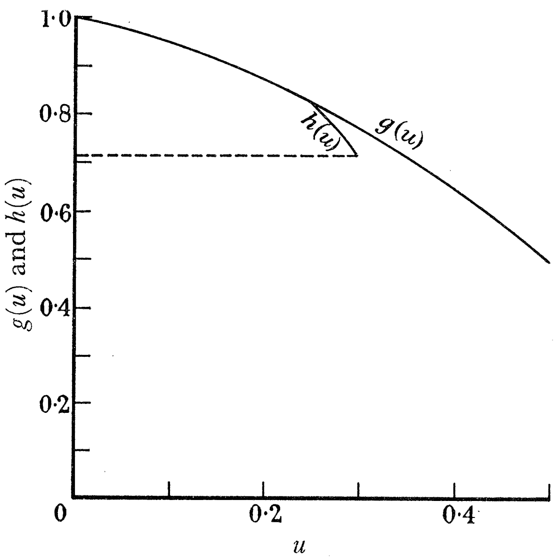

FIGURE 4. Graphs of $g(u)$ and $h(u)$. The continuous curve running from $u = 0$ to $u = 0{\cdot}5$ is $g(u)$ and the small down-projecting spur between $u = 0{\cdot}2369$ and $u = 0{\cdot}30$ is $h(u)$.

If $0 < u < 0{\cdot}3$, it is clear from figures 3 and 4 that $|\mathscr{R}m_\tau| > u$ if both

$$|a| < h(0{\cdot}3)\,(\tfrac{1}{2}u^2)^{\frac{2}{3}} \tag{56a}$$

and

$$|b|^{-1} \leqslant 0{\cdot}50. \tag{56b}$$

These two conditions are much too stringent to be necessary, but they are sufficient to assure that $|\mathscr{R}m_\tau| > u$ for both signs of τ and ω and all roots of (4). Conditions (56) are precisely conditions (6).

If σ is as small as 10^5 mho/m, then κ can be $7{\cdot}5\times10^4$ cm^2/s, and we must consider $0 < u < 0{\cdot}5$. In this case we need more stringent conditions on $|a|$ and $|b|^{-1}$ to assure $|\mathscr{R}m_\tau| > u$. Examination of figure 3 shows that one sufficient condition is

$$\frac{|a|}{g(u)\,(\frac{1}{2}u^2)^{\frac{2}{3}}} + \frac{|b|^{-1}}{0{\cdot}65} \leqslant 1,$$

or

$$\alpha + \frac{2{\cdot}0\,u^{\frac{4}{3}}}{|\beta|} \leqslant \frac{g(u)}{(2u)^{\frac{2}{3}}}. \tag{57}$$

If we write $|\tau f - \omega| = |f|$ then (57) becomes

$$A^2 + 2\nu(\zeta L)^{-\frac{4}{3}}|\omega|^{\frac{1}{3}}\kappa^{\frac{2}{3}} \leqslant 0{\cdot}63 g(u)|\omega|^{\frac{1}{3}}(\zeta L\kappa)^{\frac{2}{3}}|f|.$$

The most stringent limit is at the least value of ω, $2\pi/|\omega| = 1$ century. If we take this value for $|\omega|$, and take $L = 600$ km, $\zeta = 0{\cdot}2$, $\kappa = 7{\cdot}5\times 10^4$ cm²/s we have

$$A^2 + \frac{\nu}{6{\cdot}1\times 10^8} \leqslant 5{\cdot}2\,|\cos\theta| \tag{58}$$

as a sufficient condition that all the boundary layers be thinner than ζL when

$$0 < u < 0{\cdot}5.$$

In (58), the radial Alfvén velocity A is in centimetres per second and the kinematic viscosity ν is in centimetres squared per second.

Appendix II. The effect of the boundary layer on the magnetic field

In this appendix we examine the argument of Roberts and Scott that **B** must be constant throughout the boundary layers, and that in §3 we should take $\mathbf{b}(0) = \mathbf{0}$.

We use the notation of §3 and discuss only the case $|\beta| \gg 1$, this being the case considered by Roberts & Scott, and probably a correct description of the upper core unless there are unforeseen large pressure effects on the kinematic viscosity of iron. Throughout this appendix if z is a complex number, $z^{\frac{1}{2}}$ is the square root with positive real part. If we write $\beta = \tau'|\beta|$ then the solutions of (4) which have positive real part are

$$m_{\nu\tau} = \zeta^{\frac{1}{2}}|\beta|^{\frac{1}{2}}\left[1 - \frac{\mathrm{i}\alpha}{2\zeta^2|\beta|} + O(|\zeta\beta|^{-2})\right], \tag{59}$$

$$m_{\kappa\tau} = (-\mathrm{i})^{\frac{1}{2}(1-\tau')}\zeta^{-\frac{1}{2}}\left[1 + \frac{\mathrm{i}\alpha}{2\zeta^2|\beta|} + O(|\zeta\beta|^{-2})\right], \tag{60}$$

where α and β are α_τ and β_τ, and $\zeta_\tau = \alpha_\tau + \mathrm{i}\tau'$. Then the Ekman layers are thinner than the thinnest magnetic boundary layer by a factor of $|\beta|^{-\frac{1}{2}} \ll 1$. The thicker magnetic boundary layer is the thickest of all four layers. Its thickness is

$$(\mathscr{R}n_{\kappa\tau})^{-1} = \left|\frac{2\kappa}{\omega}\right|^{\frac{1}{2}}(\alpha^2+1)^{\frac{1}{4}}\,[(\alpha^2+1)^{\frac{1}{2}}+\alpha]^{\frac{1}{2}}, \tag{61}$$

computed by neglecting all but the leading term in (60). We have

$$\left.\begin{aligned} m^2_{\kappa\tau} &= \tau'/\zeta_\tau, \\ m^2_{\nu\tau} &= |\beta_\tau|\zeta_\tau \end{aligned}\right\} \tag{62}$$

to lowest order in $|\beta|^{-1}$.

If we write the solutions of (2) in the form

$$\left.\begin{aligned} v_\tau(z,t) &= V_\tau\,\mathrm{e}^{-n_\tau z - \mathrm{i}\omega t}, \\ b_\tau(z,t) &= B_\tau\,\mathrm{e}^{-n_\tau z - \mathrm{i}\omega t}, \end{aligned}\right\} \tag{63}$$

then we have, without approximation,

$$\left(\frac{\kappa}{\nu}\right)^{\frac{1}{2}}\frac{B_\tau}{V_\tau} = \frac{\gamma^{\frac{1}{2}}m_\tau}{m_\tau^2+i} = \frac{m_\tau^2 - i\beta}{m_\tau\gamma^{\frac{1}{2}}}.$$

When $|\beta| \gg 1$ we can use (59), (60) and (62) to obtain, correct the lowest order in $|\beta|^{-1}$,

$$\left(\frac{\kappa}{\nu}\right)^{\frac{1}{2}} \frac{B_{\kappa\tau}}{V_{\kappa\tau}} = \left(\frac{\alpha_\tau}{\zeta_\tau}\right)^{\frac{1}{2}},$$

$$\left(\frac{\kappa}{\nu}\right)^{\frac{1}{2}} \frac{B_{\kappa\tau}}{V_{\kappa\tau}} = -\mathrm{i}^{\frac{1}{2}(1+\tau')} |\beta_\tau|^{\frac{1}{2}} \zeta_\tau^{\frac{1}{2}}.$$

Therefore the solution of (2) in the form (63) which belongs to the Ekman root $m_{\nu\tau}$ has a very much smaller tangential magnetic field than the solution which belongs to the magnetic boundary layer root $m_{\kappa\tau}$. The tangential magnetic field is nearly constant in the Ekman layer, but varies in the magnetic boundary layer. Physically, this constancy in the Ekman layer is a result of that layer's thinness; it cannot carry enough current to alter **b** appreciably. More precisely, in a period $2\pi/|\omega|$ ohmic decay will eliminate any feature of **b** with length scales as short as the thickness of the Ekman layer.

It follows that when $|\beta| \gg 1$ we can carry out our discussion of **b** as if the fluid were non-viscous and slipped freely past the boundary. We examine first the limiting case $\kappa = 0$, when the magnetic boundary layer is infinitely thick. Roberts & Scott (1965) observe that a discontinuity in $\hat{\mathbf{n}} \times \mathbf{B}$ at ∂V, the boundary of the perfectly conducting fluid, requires a surface current in ∂V. As long as $\hat{\mathbf{n}} . \mathbf{B} \neq 0$, there will be a tangential Lorentz force per unit area on this current. Since the fluid is free to slip at the boundary, there is nothing to balance this Lorentz force, so in fact the fluid must move in such a way as never to produce it. That is, the fluid must move so as to make $\hat{\mathbf{n}} \times \mathbf{B}$ continuous across the boundary.

We agree with these remarks of Roberts and Scott but do not believe they imply that $\hat{\mathbf{n}} \times \mathbf{B}$ is the same just outside the core as at the top of the free stream. Consider the simplest case, a non-rotating, perfectly conducting fluid. The potential discrepancy between $\hat{\mathbf{n}} \times \mathbf{B}$ just inside and just outside ∂V is resolved, as Roberts and Scott point out, by the generation of an Alfvén wave at the boundary, which propagates into the fluid. If we consider a gross fluid motion with length scale L and period $2\pi/|\omega|$, this Alfvén wave has wavelength

$$\lambda_A = 2\pi A/|\omega|.$$

Its amplitude is given approximately by (3). If we fix ω, L_H, and the real physical fluid velocity $A\mathbf{v}(0)$, and consider configurations in which $\hat{\mathbf{n}} . \mathbf{B}$ and A become smaller and smaller, then the amplitude of the displacement **s** produced by these waves will decrease with A as follows:

$$|\mathbf{s}| \approx \frac{\lambda_A}{2\pi} |\mathbf{b}(0)| \approx \frac{A}{\omega^2 L_H} |A\mathbf{v}(0)|.$$

As $\hat{\mathbf{n}} . \mathbf{B}$ and A approach zero, the Alfvén waves continue to propagate from ∂V throughout the whole fluid volume V, but their wavelength λ_A and their amplitude $|\mathbf{s}|$ decrease linearly with A. In the limit when $\hat{\mathbf{n}} . \mathbf{B} = 0$ on all of ∂V, no Alfvén waves are generated at ∂V by the potential discontinuity in $\hat{\mathbf{n}} \times \mathbf{B}$, so it can become a real discontinuity. Then there is a surface current in ∂V, but the Lorentz force on it is normal to ∂V, because $\hat{\mathbf{n}} . \mathbf{B} = 0$. Such a surface stress can be balanced by pressure gradients and gross acceleration of the fluid.

If $\hat{\mathbf{n}} . \mathbf{B}$ on ∂V is so small that $\lambda_A \ll L$, the fluid will be permeated by very short Alfvén waves of very small amplitude. If we consider velocities and magnetic fields averaged over regions whose smallest dimension is many times λ_A, then presumably to first order in λ_A/L these small Alfvén waves will have no magnetic or dynamic effects. Their only function will be to make $\hat{\mathbf{n}}\times\mathbf{B}$ continuous at ∂V. At higher orders in (λ_A/L), their radiation pressure will transfer momentum and they will extract energy from the large-scale flow. For example, conceivably they could be an important mechanism for damping the self-gravitational oscillations of some magnetic stars.

If we start with a free-stream motion in which $\hat{\mathbf{n}}.\mathbf{B} = 0$, and add a small $\hat{\mathbf{n}}.\mathbf{B}$, so that $\lambda_A \ll L$, the averaged motion, averaged over scales much larger than λ_A, will still be the original free-stream motion, and will have a discontinuity in $\hat{\mathbf{n}}\times\mathbf{B}$ at ∂V. If $\hat{\mathbf{n}}.\mathbf{B}$ becomes so large that λ_A is comparable with L, then $\hat{\mathbf{n}}.\mathbf{B}$ will have appreciable dynamic effects on the free-stream motion, and $\hat{\mathbf{n}}\times\mathbf{B}$ will indeed be continuous across ∂V in the free-stream motion.

If the system is rotating, the two polarized Alfvén waves become two Alfvén-inertial waves with two different wavelengths. The foregoing discussion then remains valid if λ_A is replaced by the larger of these two wavelengths.

The only effect of adding a small magnetic diffusivity κ is to damp the Alfvén waves slightly as they propagate into the fluid.

When λ_A (or, in rotating systems, the longest Alfvén-inertial wavelength) becomes comparable with L, so that $\hat{\mathbf{n}}.\mathbf{B}$ is strong enough to have gross dynamic effects on the large-scale fluid motion, we can no longer speak of a magnetic boundary layer. It is still possible, however, for the Ekman layers to be much thinner than L, and for $L^2/|\omega|\kappa$ to be so large that the free stream motion outside the Ekman layer behaves as if the fluid were a perfect conductor. This is the regime of parameters which Roberts and Scott appear to have had in mind. They take the boundary layer to be the Ekman layer alone, and regard the magnetic boundary layer as part of the free stream. As we have already seen in §3, our present information about A appears to indicate that this regime is not appropriate to the Earth's core.

Incidentally, the foregoing arguments suggest that we need not require

$$(\mathscr{R}n_{\kappa\tau})^{-1} \leqslant \zeta L$$

in order to use the boundary layer theory. All we need is that $2\pi|n_{\kappa\tau}|^{-1} \leqslant \zeta L$. Then even if the Alfvén inertial waves do not damp out as they propagate into the fluid their effects on the free stream are small. In the earth's core, the second condition is more demanding than the first.

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References

Allan, D. W. & Bullard, E. C. 1958 *Rev. Mod. Phys.* **30**, 1087–1088.
Allan, D. W. & Bullard, E. C. 1966 *Proc. Camb. Phil. Soc.* **62**, 783–809.
Backus, G. E. 1958 *Ann. Phys.* **4**, 372–447.
Backus, G. E. 1966 *Arch. Rat. Mech. Anal.* **22**, 210–252.
Backus, G. E. 1967 *Geophys. J. R. Astr. Soc.* **13**, 71–101.
Bondi, H. & Gold, T. 1950 *Mon. Not. R. Astr. Soc.* **110**, 607–611.
Booker, J. 1968 *Proc. Roy. Soc.* A (in the press).

Bullard, E. C., Freedman, C., Gellman, G. & Nixon, J. 1950 *Phil. Trans.* A **243**, 67–92.
Bullard, E. C. & Gellman, H. 1954 *Phil. Trans.* A **247**, 213–278.
Bullen, K. E. 1963 *An introduction to the theory of seismology*, 3rd ed. Cambridge University Press.
Cain, J. C., Hendricks, S. J., Langel, R. A. & Hudson, W. V. 1967 Report submitted to the Working Group on the Analysis of the Geomagnetic Field Under Commission III of the International Association of Geomagnetism and Aeronomy.
Cohen, M. H. & Turnbull, D. 1959 *J. Chem. Phys.* **31**, 1164–1169.
Cox, A. & Darlymple, G. 1967 *J. Geophys. Res.* **72**, 2603–2614.
Cowling, T. G. 1957 *Magnetohydrodynamics*. New York: Interscience.
Currie, R. G. 1967 *J. Geophys. Res.* **72**, 2623–2633.
Dagley, P., Wilson, R. L., Ade-Hall, J. M., Walker, G. P. L., Haggerty, S. E., Sigurgeirsson, T., Watkins, N. D., Smith, P. J., Edwards, J. & Grasty, R. L. 1967 *Nature, Lond.* **216**, 25–29.
Elsasser, W. M. 1946 *Phys. Rev.* **69**, 106–116.
Greenspan, H. P. & Howard, L. N. 1963 *J. Fluid Mech.* **17**, 385–404.
Hide, R. 1956 In *Physics and chemistry of the earth* 1, 94–137 (ed. Ahrens, Rankama & Runcorn). London: Pergamon Press.
Hide, R. 1966 *Phil. Trans.* A **259**, 615–650.
Hide, R. & Roberts, P. H. 1960 *Rev. Mod. Phys.* **32**, 799–806.
Jeffreys, H. 1959 *The Earth*, 4th ed. Cambridge University Press.
Kahle, A. B., Ball, R. H. & Vestine, E. H. 1967 *J. Geophys. Res.* **72**, 4917–4925.
Malkus, W. V. R. 1967 *J. Fluid Mech.* **28**, 793–802.
McDonald, K. L. 1957 *J. Geophys. Res.* **62**, 117–141.
Mie, G. 1908 *Ann. Physik. Ser. 4*, **25**, 377–445.
Nachtrieb, N. H. & Petit, J. 1956 *J. Chem. Phys.* **24**, 746–750.
Roberts, P. H. 1967 *Proc. Roy. Soc.* A 300, 94–107.
Roberts, P. H. & Scott, S. 1965 *J. Geomagn. Geoelect.* **17**, 137–151.
Schechter, M. 1960 *Comm. Pure. Appl. Math.* **13**, 321–328.
Stewartson, K. 1967 *Proc. Roy. Soc.* A **299**, 173–187.
Toomre, A. 1960 (Private communication.)
Tozer, D. C. 1958 In *Physics and chemistry of the earth* **3**, 414–436 (ed. Ahrena, Rankama & Runcorn). London: Pergamon Press.

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